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CMPUT 675: Approximation Algorithms **Fall 2015**

Lecture 15&16 (Oct 20 & 22, 2015): Iterative Rounding

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15.1 Minimum Cost Bounded degree Spanning Tree

A natural generalization of minimum cost spanning tree is when we have some given degree bounds for each vertex and our goal is to find a minimum cost spanning tree statisfying the degree bounds. In other words, for a given graph $G(V, E)$ and a bound B_v for each node $v \in V$, we want to find a spanning tree with minimum cost while degree of each $v \in V$ is at most B_v (MCBDST).

Note: This problem is NP-complete since if edge costs are all 1 and all degree bounds are 2 it is the Hamiltonian path problem.

When all B_v's are equal (say k) and the graph is unweighted, the problem is finding a spanning tree of maximum degree k.

Theorem 1 (Furer & Raghavarchi '90) *There is a polynomial time algorithm that finds a spanning tree of maximum degree at most $k+1$ (if there is one of with maximum degree at most k).*

In this lecture we see the proof of the following result.

Theorem 2 (Singh & Lan '07) *For MCBDST there is a polynomial time algorithm that finds a spanning tree of cost at most optimum in which every vertex v has degree at most $B_v + 1$.*

We first formulate the problem as an IP and then consider the following LP relaxation (LP_{MBDST}):

$$\begin{aligned} (LP_{MBDST}) \quad & \min \quad \sum_{e \in E} C_e x_e \\ \text{s.t.} \quad & x(E(s)) \le |S| - 1 \qquad \forall S \subseteq V \\ & x(E(v)) = |V| - 1 \\ & x(\delta(v)) \le B_v \qquad \forall v \in W. \end{aligned} \tag{15.1}$$

Let $x(\delta(S)) = \sum_{e \in \delta_S} x_e$. The proof of following lemma is similar to the corresponding lemma we had for minimum spanning tree (using uncrossing).

Lemma 1 *Let x be a basic feasible solution to LP_{MBDST} with $x_e > 0$ for all e then there is a set $T \subseteq W$ and a laminar family $\mathcal{L}$ such that:*

1. *$x(\delta(v)) = B_v$ for each $v \in T$ and $x(E(S)) = |S| - 1$ for each $S \in \mathcal{L}$.*

2. *Vectors of* $\{\chi(E(s)) : s \in \mathcal{L}\} \cup \{\chi(\delta(v)) : v \in T\}$ *are linearly independent.*
3. $|T| + |\mathcal{L}| = |E|$

The following is our algorithm for MCBDST.

Minimum bounded degree spanning tree Algorithm

Input: Graph $G(V, E)$
Output: A minimum cost bounded degree spanning tree F.
1. $F \leftarrow \emptyset$
2. **while** $W \neq \emptyset$ **do**
3. find a basic feasible solution to LP_{MBDST} and remove any e with $x_e = 0$
4. let $v \in W$ be a node with $x(\delta(v)) = B_v + 1$
5. $W \leftarrow W - \{v\}$
6. let x be a basic feasible solution and add all $x_e > 0$ to F
7. **return** F

Figure 15.1: Algorithm Minimum cost bounded spanning tree

For now let's assume that we can always find a vertex v for line 6 of the above algorithm.

Lemma 2 *The cost of the returned solution is at most OPT.*

Proof. Note that in each iteration we remove a degree constraint. Therefore, the current LP solution is feasible for the residual LP. Thus, the cost can only go down per iteration. ∎

Lemma 3 *The returned solution is a spanning tree with each node v having degree at most $B_v + 1$.*

Proof. Since for a vertex v with degree at most $B_v + 1$ we remove the constraint on that node and we only work with edges with $x_e > 0$ in worst case the final solution has all those edges, thus the degree of v will be no more than $B_v + 1$. ∎

Theorem 3 *Let x be a basic feasible solution with $x_e > 0$ for all e, and $\mathcal{L}$ and T be the set of tight constraints (defined in Lemma 1) then if $T \neq \emptyset$, there is some vertex v such that $deg(v) \leq B_v + 1$.*

Proof. We will prove this theorem by contradiction assuming that $T \neq \emptyset$ and $\forall v \in W, deg(v) \geq B_v + 2$.

Observation 1 *If there is any edge e such that $x_e = 1$ then that edge belongs to span of $\mathcal{L}$ (set of tight constraints).*

To see this, let vertices u and v be two endpoints of the edge e. Then we define set $S = \{u, v\}$ which means that $E(S) = \{e\}$. So, $x(E(S)) = 1 = |S| - 1 = 1$ which means that constraint $x(E(s)) \leq |S| - 1$ is tight in LP_{MBDST}.

For the proof of this Theorem we will use the common argument of token assignment that we used for the proof of the previous lemmas. We assign one token to each edge. So overall we will have $|E|$ number of tokens. We will show that from this $|E|$ number of tokens each laminar family will get at least one token but there are some extra tokens which did not assign to any laminar families. We will distribute the assigned tokens from each edge $e = uv$ in a following way:

1. e will give $\frac{1-x_e}{2}$ tokens to the smallest set in $\mathcal{L}$ containing u and $\frac{1-x_e}{2}$ tokens to the smallest set in $\mathcal{L}$ containing v.

2. e will give x_e tokens to the smallest set in $\mathcal{L}$ containing both u and v.

Fist consider vertices $v \in T$. Each vertex v gets $\frac{1-x_e}{2}$ tokens from each edge incident to v. According to the assumption that $deg(v) \geq B_v + 2$ we will get the following:

$$\sum_{e \in \delta(v)} \frac{1-x_e}{2} = \frac{deg(v) - x(\delta(v))}{2} = \frac{deg(v) - B_v}{2} \geq \frac{B_v + 2 - B_v}{2} = 1$$

Which means that each vertex in T gets at least one token.

Now to show that each set in the laminar family gets at least one token we consider set S that has children $R_1, ..., R_k$ in the laminar family. Since $R_1, ..., R_k$ and S are all tight sets we have the following result:

$$x(E(S)) = |S| - 1$$

$$x(E(R_i)) = |R_i| - 1 \quad , i = 1, ..., k$$

$$x(E(S)) - \sum_{i=1}^{k} x(E(R_i)) = |S| - 1 - \sum_{i=1}^{k} x(R_i - 1).$$

Also, $x(E(S)) - \sum_{i=1}^{k} x(E(R_i)) = x(E(S) \backslash \sum_{i=1}^{k} E(R_i))$ is the number of tokens that set S will get because $E(S) \backslash \sum_{i=1}^{k} E(R_i)$ is the set of edges that are in S but not in its children. So, it will be a positive integer.

On the other hand, $|S| - 1 - \sum_{i=1}^{k} x(R_i - 1)$ cannot be zero because sets of $\mathcal{L}$ are linearly independent. As a result, each set of laminar family will get at least one token.

The size of laminar family is $|\mathcal{L}|$ and we had $|E|$ number of tokens. If each set in $\mathcal{L}$ and each vertex in T receive at least one token it means that $|\mathcal{L}| + |T| \leq |E|$. Now we need to show that some extra tokens left that we did not assign them to any set and as a result, equality does not hold.

If $V \notin \mathcal{L}$ then assume that S' is the maximal set in $\mathcal{L}$. Since graph is connected there is an edge e_1 coming out of set S'. But there is not any set in $\mathcal{L}$ that contains both endpoints of edge e_1. So, edge e_1 cannot give any sets in $\mathcal{L}$, x_e tokens. So, some extra tokens left.

If $\exists v \in V \backslash T$, then v gets $\dfrac{1-x_e}{2}$ that is unaccounted for unless $x_e = 1$. So, for each $v \in V \backslash T$ all edges must have $x_e = 1$. Hence, we can write the following equality.

$$2\chi(E(V)) = \sum_{v \in V} \chi(\delta(v)) = \sum_{v \in T} \chi(\delta(v)) + \sum_{v \in V-T} \chi(\delta(v)) = \sum_{v \in T} \chi(\delta(v)) + \sum_{v \in V-T} \sum_{e \in \delta(v)} \chi(e)$$

Since $x_e = 1$, $\chi(e)$ will be in $span(\mathcal{L})$. Since v is in $span(\mathcal{L})$, $\chi(\delta(v))$ will be in $span(\mathcal{L})$. As a result, we can write $\sum_{v \in T} \chi(\delta(v))$ as a convex combination of the other two terms which contradicts the fact that they must be linearly independent. So, $x_e \neq 1$ and there are some extra tokens that are not assigned to any laminar families. Hence, the equality does not hold and $|T| + |\mathcal{L}| < |E|$ which contradicts Lemma 1. This proves the Theorem 4.

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15.2 Survivable Network Design Problem

This problem is the generalization of steiner forest problem. In this problem we are given an undirected graph $G(V,E)$ with cost $C_e, \forall e \in E$ and connectivity requirement $r(u,v)$ for all pairs of vertices $u,v \in V$, where $u \neq v$. Also, the connectivity requirement are nonnegative integers. Our goal is to find a minimum cost set of edges $F \subseteq E$ such that for all pairs of vertices $u,v \in V$ with $u \neq v$, there are at least $r(u,v)$ edge disjoint paths connecting u and v in (V,F). Steiner forest is a spacial case of this problem where $r(u,v) = 1$ for all pairs in S_i.

Examples: If $r(u,v) = 1$ for all pairs then our problem is the Minimum Spanning Tree problem.
If $r(u,v) = 1$ for all $u,v \in T$ for a set $T \subseteq V$ then our problem is actually Steiner Tree.
If $r(u,v) \in \{0,1\}$ for all pairs then our problem is actually Steiner Forest(Generalized Steiner Tree).
If $r(u,v) = k$ for some given k, for all pairs then our problem is actually Minimum Cost k-edge-connected subgraph problem.
If $r(u,v) = 2$ for all pairs then our problem is actually Minimum Cost 2-edge-connected subgraph problem.

Definition 1 *The cut requirement function $f : 2^V \to \mathbb{Z}^+, \forall S \subseteq V$ is the largest requirement across the cut $S, \bar{S}$. In other words, $f(s) = \max_{u,v} r(u,v)$ s.t. $u \in S, v \notin S$.*

Definition 2 *We define U_e to be maximum number of copies of edge e that you can pick. In a simple graph it would be 1. But in this problem, it is possible to have a graph with multiple edges.*

Following is an LP rexation of the problem:

$$
\begin{array}{llllll}
\min & \sum_{e \in E} c_e \cdot x_e & & & & \\
\text{s.t.} & \sum_{e \in \delta(S)} x_e & \geq & f(S) & \forall S \in V & (1) \\
& 0 \leq x_e \leq U_e & & \forall e \in E & & (2)
\end{array}
$$

Definition 3 *Suppose that we have picked a set of edges so far, say $H \subseteq E$. For each set $S \subseteq V$ we define the residual cut requirement as $f'(S) = f(S) - |\delta_H(S)|$.*

Note that f' may not correspond to any connectivity requirement function $r(u,v)$.

Definition 4 *A function $f : 2^V \to Z^+$ is strongly submodular if $f(V) = 0$ and for all $A, B \subseteq V$ both of the following hold:*

1. $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$
2. $f(A) + f(B) \geq f(A - B) + f(B - A)$

Lemma 4 *Function $|\delta_G(S)|$ is strongly submodular.*

Proof. We prove that the first situation holds. The proof for second condition is very similar.

If sets $A \cap B \neq \emptyset$ or $A \subseteq B$ or $B \subseteq A$ then it is easy to check.

For the case that A and B do not cross we have Figure 15.2. We want to show that $|\delta_G(A)| + |\delta_G(B)| \geq |\delta_G(A \cup B)|) + |\delta_G(A \cap B)|$.

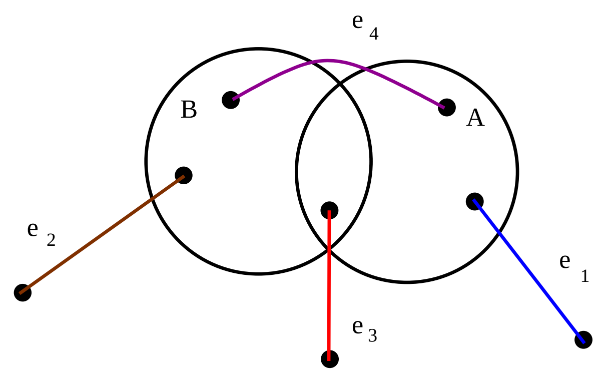

Figure 15.2: Different type of edges between crossing sets A and B.

Edges of type e_4 are contributed in both $\delta_G(A)|$ and $\delta_G(B)$ (two times in the left hand side). Edges of type e_3 contribute two times in the left hand side and two times in the right hand side. Edges of type e_1 or e_2 contribute one time in the left hand side and right hand side. So, the statement is true.

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Definition 5 *We say f is weakly super modular if at least one of the following holds. For all $A, B \subseteq V$:*

1. $f(A) + f(B) \le f(A \cup B) + f(A \cap B)$
2. $f(A) + f(B) \le f(A - B) + f(B - A)$

Lemma 5 *f and f' are weakly super modular.*

Proof.

First we will prove that $f(S) = max\{r(u,v)|u \in S, v \notin S\}$ is weakly super modular. It is clear that $f(V) = 0 = f(\emptyset)$ and $f(S) = f(V - S)$.

Also, we can observe that $f(A \cup B) \le max(f(A), f(B))$. Based on this fact we can write the following four inequalities:

$$f(A) \le max(f(A - B), f(A \cap B));$$
$$f(A) \le max(f(B - A), f(A \cup B));$$
$$f(B) \le max(f(B - A), f(A \cap B));$$
$$f(A) \le max(f(A - B), f(A \cup B));$$

For example the first inequality comes from the fact that $(A - B) \cup (A \cap B) = A$. For the second inequality we know that $f(V - A) = f(A)$ and $(B - A) \cup (V - (A \cup B)) = V - A$.

Now without loss of generality assume that $min(f(A - B), f(A \cap B), f(B - A), f(A \cup B)) = f(A \cap B)$ then by adding the first and the third inequality we have,

$$f(A) + f(B) \leq f(A - B) + f(B - A)$$

which means that f is a weakly super modular.

Then, we should prove that if f is weakly super modular then f' is weakly super modular. We know that $f'(A) = f(A) - |\delta(A)|$ and $f'(B) = f(B) - |\delta(B)|$. From Lemma 4 and first part of this Lemma we have the following:

$$(f(A) - |\delta(A)|) + (f(B) - \delta(B)| \leq (f(A - B) - |\delta(A - B)|) + (f(B - A) - |\delta(B - A)|)$$

$$f'(A) + f'(B) \leq f'(A - B) + f'(B - A).$$

As a result, f' is weakly super modular.

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15.2.1 Jain's Iterative Rounding

Following is the Jain's Iterative Rounding algorithm.

Jain's rounding

1. $H \leftarrow \emptyset$
2. $f' \leftarrow f$
3. **while** H is not feasible **do**
4. find an optimum extreme point solution (basic feasible solution) x for graph $G = (V, E - H)$ using f'
5. delete edges with $x_e = 0$
6. for each edge e with $x_e \geq \frac{1}{2}$ do
7. add $\lceil x_e \rceil$ copies of e to H
8. for every $S \subseteq V : f'(S) \leftarrow f(S) - |\delta_H(S)|$
9. **return** H

Figure 15.3: Jain's iterative rounding

Note that although the LP has exponentially many constraints we can solve this using Ellipsoid algorithm if we have a separation oracle for it. For every pair of vertices $u, v \in V(G)$ we can run a max-flow-min-cut from u to v with these x values as flows to find a min-cut between u, v. This min-cut gives you a cut S and then you need to have at least $r(u, v)$ in this cut and the capacity of the cut with these values has to be at least $r(u, v)$. If a min-cut with less capacity than $r(u, v)$ is found, then this cut represents a violated constraint.

For updating f', imagine an arbitrary iteration. Given x' we want to check feasibility using ellipsoid algorithm. We define $x_e = x'_e + e_H$, where e_H is the number of copies of e added to H so far.

Lemma 6 *Cut $(S, \bar{S})$ is violated by x' under f', if and only if it is violated by x under f.*

Proof. We know that:

$$\delta_x(S) = \delta'_x(S) + |\delta_H(S)|$$

$$f(S) = f'(S) + |\delta_H(S)|$$

So we can easily conclude that:

$$\delta_x(S) \geq f(S) \Longleftrightarrow \delta'_x(S) \geq f'(S)$$

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So, you don't need to update f' in each iteration, instead you can work with function f. The update statement in Jain's rounding is for easier description of algorithm.

Until now we have proved that Jain's algorithm can be implemented to run in polynomial time. Now we should prove that algorithm gives us a 2-approximation.

Lemma 7 (Main Lemma) *For any weakly super modular f and any basic feasible solution x, there is an edge e with $x_e \geq \frac{1}{2}$.*

For now we assume that Lemma 7 is true and we will prove that this algorithm is 2-approximation algorithm.

Theorem 4 *Jain's algorithm is 2-approximation.*

Proof. We will prove this Theorem by induction on number of iterations. Let Z_{LP} be the optimum value of original LP. Assume that $\exists e \in E$ with $x_e \geq \frac{1}{2}$ added to F and Z'_{LP} is the optimum cost of the updated LP. According to the induction hypothesis, cost of edges picked in the subsequent of iterations is at most $2Z'_{LP}$. We claim that $Z'_{LP} \leq Z_{LP} - c_e x_e$. This is because the restriction of x to the edges in $E - e$ is feasible for the residual LP. Since $x_e \geq \frac{1}{2}$ we have the following:

$$Z'_{LP} \leq Z_{LP} - c_e x_e \leq Z_{LP} - \frac{c_e}{2}$$

According to the induction hypothesis we will have:

$$cost \leq 2Z'_{LP} + c_e \leq 2(Z_{LP} - \frac{c_e}{2}) + c_e = 2Z_{LP}$$

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15.2.2 Characterization of basic feasible solutions

From now on, we prove Lemma 7 Assume that $0 < x_e < 1$ for all edges because we can delete edges with $x_e = 0$ and add those with $x_e = 1$ to the solution without any cost increase w.r.t optimum. Let $m = |E|$ be the number of edges after deleting 0 and 1 values, i.e. the size of the totally fractional solution. For every set $F \subseteq E$ we use $x(F)$ to denote the sum of the x_e values for $e \in F$.

Definition 6 *A set S is called tight if $x(\delta(S)) = f(S)$.*

Definition 7 *For each set $F \subseteq E$ the characteristic vector is defined as $\chi_F \in \{0,1\}^{|E|=m}$, such that $\chi_F(e) = \begin{cases} 1 & if\ e \in F \\ 0 & otherwise \end{cases}$*

Lemma 8 *For any basic feasible solution x that is totally fractional (i.e. $0 < x < 1$) there is a collection of $m = |E|$ tight sets $\mathcal{L}$ s.t.*

1. *$\mathcal{L}$ is Laminar.*
2. *characteristic vectors of $\delta(S)$ ($S \in \mathcal{L}$) are linearly independent.*

To prove this lemma we need to prove some other statements first. Let x be a basic feasible solution and τ be a collection of m tight constraints corresponding to this solution. (note that they are all linearly independent)

Definition 8 *Span of τ is the vector space generated by these m linearly independent vectors, i.e. $\{\chi_{\delta(S)} : S \in \tau\}$. So, we will have $span(\tau) = \mathbb{R}^m$*

Let $\mathcal{L}$ be a maximal laminar family that is a subset of τ. If $|\mathcal{L}| = m$ then we have what we need in Lemma 8. Otherwise, we show there is a tight set S which can be added to $\mathcal{L}$ while increasing the span of $\mathcal{L}$ and keeping it Laminar.

Lemma 9 *Let τ be a collection of m tight constraints corresponding to basic feasible solution and they are all linearly independent. If $A, B \in \tau$ and are crossing (tight and linearly independent), then one of following two holds:*

1. *$A \cup B$ and $A \cap B$ are tight and $\chi(\delta(A)) + \chi(\delta(B)) = \chi(\delta(A \cap B)) + \chi(\delta(A \cap B))$*
2. *$A - B$ and $B - A$ are tight and $\chi(\delta(A)) + \chi(\delta(B)) = \chi(\delta(A - B)) + \chi(\delta(B - A))$*

Remark: This lemma can be used to uncross two sets A and B that cross by replacing them with $A \cup B$ and $A \cap B$ or $A - B$ and $B - A$.

Proof. We know that f is weakly super-modular. Suppose that the second condition holds: $f(A) + f(B) \leq f(A - B) + f(B - A)$ (the proof when the first condition holds is similar). Because $A, B \in \mathcal{T}$ both are tight we must have $x(\delta(A)) = f(A)$ and $x(\delta(B)) = f(B)$.

So $x(\delta(A)) + x(\delta(B)) = f(A) + f(B) \leq f(A - B) + f(B - A)$. Because x is feasible $x(\delta(A - B)) \geq f(A - B)$ and $x(\delta(B - A)) \geq f(B - A)$; thus $x(\delta(A - B)) + x(\delta(B - A)) \geq f(A - B) + f(B - A)$. Combining these two yields $x(\delta(A)) + x(\delta(B)) \leq x(\delta(A - B)) + x(\delta(B - A))$.

We show that this holds with equality by the same technique as Lemma 4. Consider the four edge types in Figure 15.2. Edge e_3 contributes twice to the left hand side but edges e_1, e_2 and e_4 contribute equally to both sides. Therefore the inequality holds in other direction too; so we must have equality.

Because this holds with equality and we also have $x(\delta(A - B)) \geq f(A - B)$ and $x(\delta(B - A)) \geq f(B - A)$ (because x is a feasible solution) we must have $x(\delta(A - B)) = f(A - B)$ and $x(\delta(B - A)) = f(B - A)$. Also, there is only one type of edge, namely e_3 in the figure, that does not contribute to both sides equally; since we have $x(\delta(A)) + x(\delta(B)) = x(\delta(A - B)) + x(\delta(B - A))$, these edges must have value 0, i.e. don't exist. Thus $\chi(\delta(A)) + \chi_{\delta(B)} = \chi_{\delta(A-B)} + \chi_{\delta(B-A)}$, as wanted. ∎

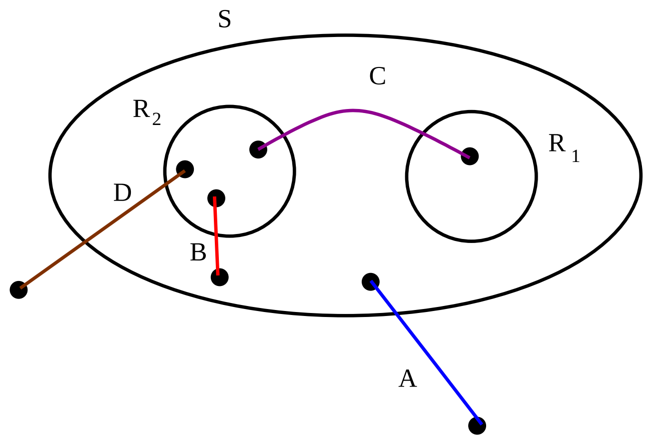

Figure 15.4: Diferent type of edges between R_1 and R_2.

We now return to prove lemma 8.

Proof. for Lemma 8

Assume $|\mathcal{L}| < m$. There must be a tight set S s.t. $\chi_{\delta(S)} \in span(\tau)$ but $\chi_{\delta(S)} \notin span(\mathcal{L})$. Pick such S that crosses the fewest number of sets in $\mathcal{L}$. Suppose that S crosses $S' \in \mathcal{L}$. Apply Lemma 9 to S, S'. Either $S \cap S'$ and $S \cup S'$ or $S - S'$ and $S' - S$ are tight and at least one of the following holds:

1. $\chi_{\delta(S)} + \chi_{\delta(S')} = \chi_{\delta(S \cup S')} + \chi_{\delta(S \cap S')}$
2. $\chi_{\delta(S)} + \chi_{\delta(S')} = \chi_{\delta(S - S')} + \chi_{\delta(S' - S)}$

We know that $S \notin \mathcal{L}$ and $S' \in \mathcal{L}$. If from the two situation mentioned above the first one is true, then at least one of $S \cup S'$ and $S \cup S'$ are not in $\mathcal{L}$ and if the second statement is true, then at least one of $S - S'$ and $S' - S$ are not in $\mathcal{L}$. In either case it is easy to show that for example $S \cup S' \notin \mathcal{L}$ crosses fewer sets in $\mathcal{L}$ than S'.

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Now we can prove the main lemma 7 by contradiction. We assume that for all $e \in E, 0 < x_e < \frac{1}{2}$. At the end we will get a contradiction to the fact that $|\mathcal{L}| = |E|$. We will assign one token to each edge. So overall we will have $|E|$ number of tokens.

We will distribute the assigned tokens to each edge $e = uv$ in a following way:

1. e will give x_e tokens to a smallest set in $\mathcal{L}$ containing u and x_e tokens to a smallest set in $\mathcal{L}$ containing v.
2. e will give $1 - 2x_e$ tokens to a smallest set in $\mathcal{L}$ containing both u and v.

Now to show that each set in laminar family get at least one token, we consider set S that contains two children R_1 and R_2. Assume that we have four different type of edges between these sets which are shown in Figure 15.4.

By using the two rules that are mentioned earlier for distributing tokens, the total number of tokens that set S will get is the following:

$$x(A) + x(B) + |B| - 2x(B) + |C| - 2x(C) \geq 0$$

$$|C| + |B| + x(A) - 2x(C) - x(B) \geq 0$$

Since this is the number of tokens that set S will get, the left hand side is an integer greater than or equal to zero.

Also, we have the following tight constraints for R_1, R_2 and S.

$$\begin{aligned} x(\delta(S)) &= f(S) \\ -x(\delta(R_1)) &= -f(R_1) \\ -x(\delta(R_2)) &= -f(R_2) \end{aligned}$$

We substitute the values for $\delta(S)$, $\delta(R_1)$ and $\delta(R_2)$ based on different types of edges A, B, C and D. Then we sum up these 3 equality. As a result we get the following result:

$$\begin{aligned} x(\delta(S)) - x(\delta(R_1)) - x(\delta(R_2)) = \\ x(D) + x(A) - x(C) - x(D) - x(B) - x(c) = \\ x(A) - 2x(C) - x(B) = f(s) - f(R_1) - f(R_2) \end{aligned}$$

Recall that sets of $\mathcal{L}$ are linearly independent. So, the right hand side of last equality can not be zero. So, $x(A) - 2x(C) - x(B)$ is integer and is greater than 0 which means that each set will get at least one token.

The size of laminar family is $\mathcal{L}$ and we had $|E|$ number of tokens. If each set receives at least one token it means that $|\mathcal{L}| \leq |E|$. Now we need to show that some extra tokens left that we did not assign them to any set.

Assume that S being the maximal set in $\mathcal{L}$. Recall that $\mathcal{L}$ does not contain V because $\mathcal{L}$ is a family of tight sets. Since graph is connected there is an edge e_1 coming out of set S. But there isn't any set in $\mathcal{L}$ that contains the whole of edge e_1. So, edge e_1 cannot give any sets in $\mathcal{L}$, $1 - 2x_e$ tokens. So, there are some tokens left that are unaccounted for unless $x_e = \dfrac{1}{2}$. But we assumed that $\forall e \in E, 0 < x_e < \dfrac{1}{2}$.

Hence, the equality does not hold and $|\mathcal{L}| < |E|$ which contradicts Lemma 8. This proves Lemma 7.