CMPUT 675: Randomized Algorithms **Fall 2005**

Lecture 13: Oct 20

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13.1 st-Connectivity

Consider the following problem. We are given an undirected graph, $G(V, E)$ with vertices $s, t \in V$. The question: is there a path from s to t in graph G? This problem can be easily solved in $O(n+m)$ using Depth First Search or Breadth First Search. The question is, can we solve this in $O(\log n)$-space? Note that the each of the simple traversal algorithms require $O(\log^2 n)$ space.

Definition 13.1 *The class of languages A such that there is a deterministic Turing Machine (TM), M, determining A using only* log*-space is denoted by L.*

Class RL is the randomized version of L, which has the same relation to L as RP does to P.

Definition 13.2 *RL is the set of languages A such that there is a probabilistic* log*-space TM, M such that on input x:*

$$\Pr[M\,acceptsx] = \begin{cases} \geq \frac{1}{2} & \textit{if } x \in A, \\ 0 & \textit{otherwise} \end{cases}$$

One of the big problems in computational complexity: is $RL = L$?

Theorem 13.3 *st-connectivity $\in RL$.*

Proof: Simulate a simple random walk from s for $2n^3$ steps. Since $C_{st} = 2m \cdot R_{st} \leq 2mn \leq n^3$, using Markov's inequality: $\Pr[failure] \leq 1/2$. ■

The natural question is, can we do this deterministically? i.e. is there a deterministic log-space TM to decide st-connectivity? Here is one approach to derandomize this algorithm (but it gives non-uniform TM's, i.e. one TM for each input size n).

On a d-regular graph, for each vertex, label the end of each connecting edge with a random number from $[1..d]$. The labels assigned to an edge from its two end-points may or may not be the same. Consider a sequence $S = S_1, S_2, ..., S_i, ...$ with each $S_i \in 1, ..., d$ and a starting vertex v. This defines a walk in the graph as follows: start at v and at each step i, take the edge out of the current vertex with label s_i and follow that edge to get to the next vertex.

Definition 13.4 *A sequence S is a traversal if the walk corresponding to S visits all the vertices.*

Definition 13.5 *A sequence S is a universal traversal for a class of labeled graphs if it is a traversal for all of them.*

Given n, if we have a universal traversal sequence with size $poly(n)$, we can embed it into the states of a TM. Then for any given instance of st-connectivity of size n, we can run this TM (with a fixed labeling for every vertex). However, as we said earlier, since we have a different TM for each input size, we only get non-uniform class of TM'.s

Let $\mathcal{G}_n$ be a family of d-regular labeled n-vertex graphs, $U(\mathcal{G}_n)$ be the length of the shortest traversal for all graphs in $\mathcal{G}_n$, and $R(\mathcal{G}_n)$ be the maximum resistance between any pair of vertices in any graph in $\mathcal{G}_n$.

Theorem 13.6 $U(\mathcal{G}_n) \leq O(mR(\mathcal{G}_n)\log(n|\mathcal{G}_n|))$.

Proof: We use the probabilistic method. Consider a random walk of length $8m \cdot R(\mathcal{G}_n)\log(n|\mathcal{G}_n|)$ on a graph $G \in \mathcal{G}_n$ from some vertex $u \in G$. Break the walk into sections of length $8m \cdot R(\mathcal{G}_n)$ each. Recall that for any two vertices u and v: $C_{uv} = 2mR_Puv$. The probability that we do not visit vertex v is $\leq 1/4$ for any window of size $8m \cdot R(\mathcal{G}_n)$. It follows that:

$$\Pr[v \text{ is not visited at all}] \leq (\frac{1}{4})^{\log(n \cdot |\mathcal{G}_n|)} = \frac{1}{(n|\mathcal{G}_n|)^2}.$$

Summing over all possible graphs and all possible choices of v: the probability that the walk does not visit all vertices is ¡ 1. Thus, there is a universal traversal of this length. ■

Lemma 13.7 *The number of d-regular labeled graphs of size n is* $(nd)^{O(nd)}$.

The proof of this lemma is left as an exercise.

Corollary 13.8 *The length of shortest traversal sequence for an n-vertex d-regular graph is* $O(n^3 d \log n)$.

Proof: Any d-regular graph has diameter $O(\frac{n}{d})$. Therefore: $R(G) \leq \frac{n}{d}$. Since $m = nd/2$,

$$U(G) \leq \frac{n}{d}\frac{dn}{2}\log nd^{O(nd)} \in O(n^3 \log n)$$

■

Very recently, Reingold (STOC'05) showed that st-connectivity is indeed in L.

13.2 Markov chain terminology

Definition 13.9 *A discrete stochastic process* $X_0, X_1, ...$ *is a Markov chain if:*

$$\begin{aligned} \Pr[X_t = a_t | X_0 = a_0 ... X_{t-q} = a_{t-1}] &= \Pr[X_t = a_t | X_{t-q} = a_{t-1}] \\ &= P_{a_{t-1}, a_t} \end{aligned}$$

The matrix $P_{ij} = 1$ *is called the transition matrix.*

$$\sum P_{ij} = 1 \qquad \textit{Where } 0 \leq P_{ij} \leq 1. \tag{13.1}$$

A Markov chain is memoryless, the probability that we are in state j in the next step only depends on our current state and not on the history of the chain. Let q_i^t be the probability that we are at state i at time t. Then $q^t = (q_0^t, q_1^t, ..., q_n^t)$ is the probability vector for time t. It is easy to see that $q_1^t = \sum_j q_j(t-1)p_{ji} \rightarrow q^t = q^{t-1}p$ by induction $\rightarrow q^t = q_0 p^t$ where q_0 is the initial distribution of states.

Definition 13.10 *The stationary distribution of a Markov chain is the probability distribution* Π *such that:* $\Pi = \Pi P$*, i.e. if a Markov chain is in stationary distribution at step* t *then it remains in the stationary distribution at step* $t+1$.

Stationary distribution is a steady state or equilibrium of Markov chain.

Definition 13.11 *State* i *is accessible from* j *if for some time* $t \geq 0$, $P_{ij}^t > 0$.

If both are accessible, then $i \leftrightarrow j$ *and we say they communicate. This function is an equivalence relation because the function is reflexive, symmetric, and transitive.*

Definition 13.12 *A Markov Chain MC, is irreducible if all states belong to the same communication class (i.e. the nodes corresponding to those states in the graph representation of the M.C. belong to one strongly connected component).*

Let r_{ij}^t be the probability that starting from i the first transition to j happens in time t. Then:

$$r_{ij}^t = \Pr[X_t = j \text{ and } X_s \neq j \text{ where } i \leq s \leq t-1 | X_0 = i]$$

Definition 13.13 *A state is persistent if* $\sum_{t \geq i} r_{if}^t = 1$. *A state is transient if* $\sum_{t \geq i} r_{if}^t < 1$.

Let h_{ij} be the expected number of steps to reach j from i.

$$h_{ij} = \sum_{t \geq 1} t r_{ij}^t.$$

Note that, if a state i is transient, then $h_{ii} = \infty$

Definition 13.14 *A persistent state with* $h_{ii} = \infty$ *is a null-persistent state.*

Example: Consider the following M.C. Each state corresponds to an integer. Starting from 1, the probability of going to i from $i+1$ is $i/(i+1)$ and with probability $1/(i+1)$ we go back to 1. That is: $P_{i,i+1} = \frac{i}{i+1}$ and $P_{i,1} = \frac{1}{i+1}$.

Pr[not returning to zero after t steps] $= \prod_{i=1}^t \frac{i}{i+1} = \frac{1}{t+1}$

Therefore, the probability that this walk never goes back to 0 is 0. In other words zero is a persistent state. However:

$$r_{1,1}^t = \frac{1}{t} \cdot \frac{1}{t+1} = \frac{1}{t(t+1)}.$$

And

$$h_{1,1} = \sum_{t>1}^{\infty} t \cdot r_{1,1}^{t} = \sum_{t \geq 1} \frac{1}{t+1} \longrightarrow \infty.$$

So state 0 is null-persistent. Note that this is not the case with finite MC's, i.e. having infinite number of states is necessary for null-persistent states.

Lemma 13.15 *if M is a finite M.C. then*

- *M has at least one persistent state,*
- *All persistent states are non-null*

Definition 13.16 *A state j is periodic if there is a $T > 1$ such that* $\Pr[X = j | X_t = j] = 0$ *unless S is a factor of T. If a state is not periodic, it is aperiodic. A M.C. is aperiodic if all the states of the MC are aperiodic.*

Definition 13.17 *Ergodic means aperiodic and non-null persistent.*

Theorem 13.18 (Fundamental Theorem of Markov Chains) *Any finite, irreducible, ergodic MC satisfies*

- *H has a unique stationary distribution P_i*
- $\forall c, j : \lim_{t\to\infty} P_{j,i}^{t}$ *exists and is independent of j*
- $\prod_i = \lim_{t\to\infty} P_{j,i}^{t} = \frac{1}{h_{ij}}$