Lecture 24: Graph Algorithms

Agenda:

- Graph notions (recall)
- Graph representations (recall)
- Graph traversals
  - Breadth-first search (BFS)

Reading:

- Textbook pages 525 – 539

Review of some notions:

- (simple, undirected) graph $G = (V, E)$
  - vertex set V
  - edge set E
    - ∗ an edge is an unordered pair of two distinct vertices
- Notions:
  - adjacent (vertex – vertex, edge – edge)
  - incident (vertex – edge)
  - neighborhood of a vertex
  - degree of a vertex — size of its neighborhood
  - path (vertex – vertex)
  - simple path
  - connected (every pair of vertices is connected via a path)
  - subgraph $G' = (V', E')$ of $G = (V, E)$
    - ∗ it is a graph, $V' \subseteq V$, and $E' \subseteq E$
  - connected component (maximal connected subgraph)
  - vertex-disjoint simple paths
  - biconnected graph
    - ∗ connected, and every pair of vertices are connected via two vertex-disjoint (simple) paths
  - biconnected component — maximal biconnected subgraph

More notions:

- Notions on simple, undirected graphs:
  - cycle — a path with two ending vertices collapsed
  - simple cycle
  - acyclic graph — a graph containing no cycles — also called *forest*
  - tree — connected forest
  - complete graph ($|E| = \dfrac{|V| \times (|V| - 1)}{2}$)
    every pair of vertices are adjacent
  - induced subgraph on a subset of vertices, say $U \subset V$
    $(U, E[U])$, where $E[U] = \{(v_1, v_2) : (v_1, v_2) \in E \&\& v_1, v_2 \in U\}$
  - clique (subset of vertices) — the induced subgraph is complete
  - independent set (of vertices) — the induced subgraph contains no edge
  - vertex coloring — adjacent vertices get distinct colored
- Graph variants:
  - multigraph (remove "simple")
  - digraph (remove "undirected")
  - multi-digraph (remove "simple" and "undirected")
  - edge-weighted graph (every edge has a weight)

Additional notions on graph variants:

- Rooted tree:
  - directed tree
  - one vertex as the root
  - can define the child-parent relationship
  - we have seen this (forest of rooted trees, in DSUF)
- When the base graph is directed, path/cycle is also directed
- Directed acyclic graph — DAG

Two representations:

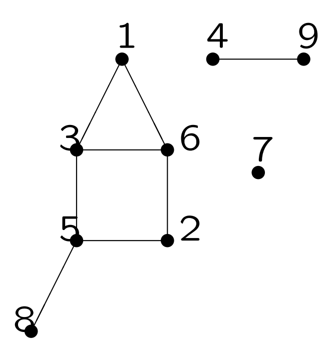
- Adjacency lists: for example,

1:	3	6	
2:	5	6	
3:	1	5	6
4:	9		
5:	2	3	8
7:	1	2	3
7:			
8:	5		
9:	4		

- Adjacency matrix: for example,

	1	2	3	4	5	6	7	8	9
1			*			*			
2					*	*			
3	*				*	*			
4									*
5		*	*					*	
6	*	*	*						
7									
8					*				
9				*					

They both describe the following graph (graphical view):



Graph traversal:

- The most elementary graph algorithm:
  - goal: visit all vertices, by following all edges in some order
  - *e.g.*, maze traversal
  - the most common graph traversal with a list storing "waiting" vertices
    1. FIFO list (queue) — breadth first search
    2. LIFO list (stack) — depth first search
    3. recursive — depth first search
- Applications:
  - finding connected components
  - determining if the graph is 2-colorable
  - finding an odd cycle, if exists
  - computing pairwise (unweighted) distance (BFS)
  - finding biconnected components (DFS)
  - finding strongly connected components (DFS)

Breadth First Search (BFS):

- Input: simple undirected graph $G = (V, E)$ and start vertex s
- Output: distance (smallest number of edges) from s to each reachable vertex
  (in a same connected component, if G is not connected)
- Pseudocode:

```
procedure BFS(G, s)        **G = (V, E), s ∈ V start vertex

for each v ∈ V − s do
    c[v] ← WHITE           **unknown yet
    d[v] ← ∞               **distance from s
    p[v] ← NIL             **predecessor
Q ← ∅                      **waiting vertex queue
enqueue(Q, s)
c[s] ← GRAY                **in queue Q
d[s] ← 0
while Q ≠ ∅ do
    u ← dequeue(Q)
    for each v ∈ Adj[u] do
        if c[v] = WHITE then
            c[v] ← GRAY
            d[v] ← d[u] + 1
            p[v] ← u
            enqueue(Q, v)
    c[u] ← BLACK           **visited
```

- An example:

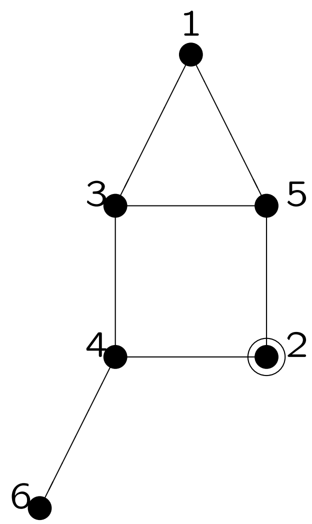
  $V = \{1, 2, 3, 4, 5, 6\}$

  $E = \{\{1, 3\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 6\}\}$

  $s = 2$

BFS example:

- $V = \{1, 2, 3, 4, 5, 6\}$
  $E = \{\{1, 3\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 6\}\}$
  $s = 2$



Adjacency lists:

1:	3	5	
2:	4	5	
3:	1	4	5
4:	2	3	6
5:	1	2	3
6:	4		

BFS example:

	1	2	3	4	5	6	Q
color	W	G	W	W	W	W	$\{2\}$
distance	∞	0	∞	∞	∞	∞	
parent	NIL	NIL	NIL	NIL	NIL	NIL	
color	W	B	W	G	G	W	$\{4, 5\}$
distance	∞	0	∞	1	1	∞	
parent	NIL	NIL	NIL	2	2	NIL	
color	W	B	G	B	G	G	$\{5, 3, 6\}$
distance	∞	0	2	1	1	2	
parent	NIL	NIL	4	2	2	4	
color	G	B	G	B	B	G	$\{3, 6, 1\}$
distance	2	0	2	1	1	2	
parent	5	NIL	4	2	2	4	
color	G	B	B	B	B	G	$\{6, 1\}$
distance	2	0	2	1	1	2	
parent	5	NIL	4	2	2	4	
color	G	B	B	B	B	B	$\{1\}$
distance	2	0	2	1	1	2	
parent	5	NIL	4	2	2	4	
color	B	B	B	B	B	B	$\emptyset$
distance	2	0	2	1	1	2	
parent	5	NIL	4	2	2	4	

BFS example:

- Adjacency lists:

1:	3	5	
2:	4	5	
3:	1	4	5
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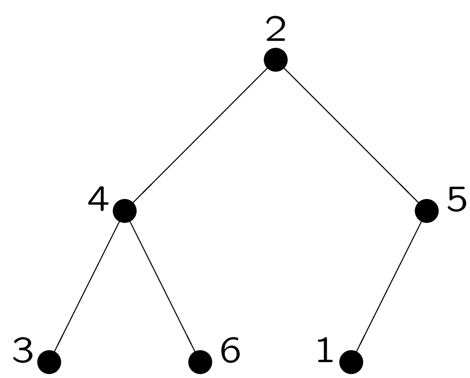
5:	1	2	3
6:	4		

- BFS tree:



Notes:

- root is the start vertex s
- parent of x is predecessor $p[x]$
- left-to-right child order *depends* on neighbor ordering (in $Adj[u]$)

BFS analysis:

- $n = |V|$, $m = |E|$
- Handshaking Lemma: $\sum_{v \in V} \texttt{degree}(v) = 2m$
- Analysis:
  - each vertex enqueued exactly once: WHITE $\rightarrow$ GRAY
  - each vertex dequeued exactly once: GRAY $\rightarrow$ BLACK
  - running time:
    1. adjacency list representation:
       $\Theta(n + \sum_{v \in V} \texttt{degree}(v)) = n + 2m) = \Theta(n + m)$
    2. adjacency matrix representation:
       $\Theta(n + \sum_{v \in V} n = n + n^2) = \Theta(n^2)$
  - space complexity:
    1. adjacency list representation:
       $\Theta(n + \sum_{v \in V} \texttt{degree}(v)) = n + 2m) = \Theta(n + m)$
    2. adjacency matrix representation:
       $\Theta(\sum_{v \in V} n = n^2) = \Theta(n^2)$
- BFS product:
  1. every s-to-v shortest path (tracing the parents)
  2. putting these paths together forms the BFS tree
- Warning: vertices in other connected components wouldn't be discovered !!!

  EXERCISE: modify the pseudocode to discover ALL vertices

Have you understood the lecture contents?

well	ok	not-at-all	topic
☐	☐	☐	graph representation recall
☐	☐	☐	graph notions
☐	☐	☐	graph variants?
☐	☐	☐	breadth first search execution
☐	☐	☐	breadth first search analysis