Graph Colouring via the Discharging Method

by

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Abstract

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In this thesis we study two colouring problems on planar graphs. The main technique we use is the Discharging Method, which was used to prove the Four Colour Theorem.

The first problem we study is a conjecture of Steinberg which states that every planar graph without 4 and 5-cycles is 3-colourable. Erdös relaxed this conjecture by asking if there exists a k such that every planar graph without cycles of size in $\{4, \ldots, k\}$ is 3-colourable. Abbott and Zhou [1] answered the question of Erdös by showing that such a k exists and can be as small as 11, i.e. any planar graph without cycles of size in $\{4, \ldots, 11\}$ is 3-colourable. This result was improve by Borodin [15] to $k = 10$, and by Borodin [14] and by Sanders and Zhao [49] to $k = 9$. We improve these results by two steps.

First we reduce k down to 8. That is, we show every planar graph without cycles of size in $\{4, \ldots, 8\}$ is 3-colourable. This theorem is constructive and yields an $O(n^2)$ time algorithm for 3-colouring such graphs.

Then we improve this result one step further, by showing that every planar graph without cycles of size in $\{4, \ldots, 7\}$ is 3-colourable. This theorem too is constructive and yields an $O(n^3)$ time 3-colouring algorithm for such graphs.

The second problem is the problem of colouring the squares of planar graphs. Equivalently, it is the problem of colouring the vertices of a planar graph in such a way that vertices at distance at most 2 from each other get different colours. This is also known as distance-2-colouring. Wegner in 1977 conjectured that, for every planar graph G with maximum degree $\Delta \geq 8$, the minimum number of colours required in any distance-2-colouring of G is at most $\lfloor \frac{3}{2}\Delta \rfloor + 1$. This conjecture, if true, would be the best possible upper bound for the number of colours needed, in terms of Δ. The previously best known bound for this quantity is $\lceil \frac{9}{5}\Delta \rceil + 1$, for graphs with $\Delta \geq 47$, by Borodin et al. [16, 17]. We improve this result by showing that $\lceil \frac{5}{3}\Delta \rceil + O(1)$ colours are enough for a distance-2-colouring of a planar graph with maximum degree Δ. We also provide a better bound for large values of Δ. Then we generalize this result to $L(p,q)$-labelings of planar graphs. An $L(p,q)$-labeling of a graph G is an assignment of integers from $\{0,\ldots,k\}$ to the vertices of G such that every two adjacent vertices in G receive integers that are at least p apart and every two vertices at distance two from each other receive integers that are at least q apart. The minimum k for which there is an $L(p,q)$-labeling of G is denoted by $\lambda_q^p(G)$. We prove that for any planar graph G: $\lambda_q^p(G) \leq q\lceil \frac{5}{3}\Delta \rceil + O(p+q)$. This improves the previously known bound of $(4q-2)\Delta + O(p+q)$, by Van den Huevel and McGuinness [57]. All these results are constructive; we provide efficient algorithms for distance-2-colouring of planar graphs with at most $\lceil \frac{5}{3}\Delta \rceil + O(1)$ colours and for $L(p,q)$-labeling of planar graphs using only $q\lceil \frac{5}{3}\Delta \rceil + O(p+q)$ colours.

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Credits

This thesis is based on the following four papers: "A bound on the chromatic number of the Square of a Planar Graph" [41] (joint with M. Molloy), "Frequency Channel Assignment on Planar Networks" [42] (joint with M. Molloy), "The Three Colour Problem for Planar Graphs" [48], and "Planar Graphs Without Cycles of Length From 4 to 7 Are 3-Colourable" [18] (joint with O. V. Borodin, A. N. Glebov and A. Raspaud).

Contents

Chapter 1

Preliminaries

The Four Colour Problem (4CP) is perhaps one of the easiest combinatorial problems to state. This seemingly simple, yet extremely difficult, problem was the most challenging problem in graph theory for well over a century. Many parts of graph theory, in particular the branch of graph colouring, grew up around this problem as byproducts of the efforts researchers put into solving this problem. One of the techniques which was specifically developed to solve the 4CP (and which we use extensively in this thesis), is the Discharging Method. Over the past few decades, this technique has been used to nail down dozens of other problems. However, there are many problems left open, for which this technique seems to be the most promising tool to apply.

In this thesis, we address two of these problems, which are in the same family as the 4CP; both of them are problems on colourings of vertices of planar graphs and were introduced almost around the same time as the 4CP was solved. Since then, some partial results have been provided on each of them, using the Discharging Method. The improvements we obtain also use the Discharging Method. So we begin with a short history of the 4CP and the development of the Discharging Method. This is done in the next chapter. Before that, we have to define some common notation used throughout the thesis. This is done in this chapter (in the next section), followed by an overview of the thesis. Some more specific terms are defined throughout the thesis, when they are needed.

1.1 Notation and Definitions

For a graph G, we denote the vertex set and edge set by $V(G)$ and $E(G)$ (or simply V and E), respectively. All graphs are assumed to be finite, undirected, and simple (without loops or multiple edges) unless otherwise stated. A *cut-vertex* in a graph G is a vertex v whose removal increases the number of connected components of G. A maximal connected subgraph of G that has no cut-vertex is a *2-connected component* or a *block* of G. A graph G is *2-connected* if it has no cut-vertices. A *cut-edge* (or *bridge*) is an edge whose removal increases the number of connected components of G.

The degree of a vertex $v \in V(G)$, denoted by $d_G(v)$, is the number of edges incident with it. The maximum and minimum degree of a graph G are denoted by $\Delta(G)$ and $\delta(G)$ (or simply Δ and δ), respectively. If the degree of v is i, at least i, or at most i we call it an i-vertex, a $\geq i$-vertex, or a $\leq i$-vertex, respectively. By $N_G(v)$, we mean the open neighbourhood of v in G, which contains all those vertices that are adjacent to v in G. The closed neighbourhood of v, which is denoted by $N_G[v]$, is $N_G(v) \cup \{v\}$. We usually use $N(v)$ and $N[v]$ instead of $N_G(v)$ and $N_G[v]$, respectively. The *square* of a graph G, denoted by G^2, is the graph on the same vertex set as G, in which two vertices are adjacent iff their distance in G is at most two. In other words, G^2 is obtained from G by adding the edges between the vertices at distance two of each other.

A graph G is *embedded* on a surface $\mathcal{S}$ if its vertices are mapped to distinct points of $\mathcal{S}$, and edges are mapped to simple curves in $\mathcal{S}$ connecting its vertex-points. Moreover, no two edge-curves share a point in $\mathcal{S}$ except possibly a common vertex-point in G. A *face* of an embedding of G is a connected component of the surface $\mathcal{S}$ after deleting the graph G. A graph G is *planar* if it has an embedding on the sphere. Since a plane is topologically equivalent to a sphere with a hole in it, every planar graph is also embeddable on a plane, and the face containing the hole is called the *external* or *outside* face. For an embedding of a planar graph G, the set of faces of G is denoted by $F(G)$, or simply F. Through a slight abuse of notation, when no confusion is possible, we say vertices of a face f to refer to the vertices that are on the boundary of face f, i.e. the vertices that are incident with f. For every face f the *size* or *length* of f, denoted by $|f|$, is the number of edges in f, with bridges (cut-edges) counted twice. A face is called an i-face, $\leq i$-face, or a $\geq i$-face if the size of f is i, at most i, or at least i, respectively. A planar graph G is called a *triangulation* if every face of G has size 3. Euler's formula (given below) plays a key role in our proofs, and in general, in the proofs of problems on planar graphs that use the Discharging Method.

> **Euler's Formula:** For any planar graph G with vertex set V, edge set E, and face set F: $|V| - |E| + |F| = 2$.

A (proper) *vertex colouring* of a graph G is a function $\varphi : V(G) \longrightarrow C$, where C is a set of colours, such that no two adjacent vertices receive the same colour. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum $|C|$ for which G has a vertex-colouring. A graph G is called *k-chromatic* if $\chi(G) = k$. A k-chromatic graph G is called *k-critical* if for any proper subgraph G' of G: $\chi(G') < k$. Note that any k-chromatic graph can be transformed into a k-critical graph by removing some vertices and/or edges from it.

1.2 Overview

The main contributions of this thesis are improvements on two different conjectures regarding colouring problems for planar graphs.

The first problem, which is the primary subject of Chapters 3 and 4, is on the colouring of planar graphs without cycles of size in $\{4, \ldots, k\}$. It is a long-standing conjecture by Steinberg that any planar graph without cycles of size in $\{4, 5\}$ is 3-colourable. We will show in Chapter 3 that planar graphs without cycles of size in $\{4, \ldots, 8\}$ are 3-colourable. This proof uses the Discharging Method.

In Chapter 4, we improve the result of Chapter 3 one step further, by showing that even in the presence of cycles of size 8, the planar graph is still 3-colourable. The proof technique used here is different than that of Chapter 3, although it also involves the Discharging Method.

In Chapter 5 we study the problem of colouring the square of a planar graph. We obtain an upper bound for the chromatic number of the square of a planar graph in terms of its maximum degree, Δ. This result tightens the asymptotic gap between the best possible upper bound and the best known upper bound. We also show how this proof can be applied to a more general setting of colouring, known as λ-colouring, and obtain a similar bound in terms of Δ, which improves all previously known bounds. Finally, we discuss some possible steps that would have to be taken to further improve these results, asymptotically.

Chapter 6 contains the concluding remarks and discussions about possible future directions.

Chapter 2

What is the Discharging Method?

In this chapter we explain, by the means of some examples, how the Discharging Method works. As we mentioned in the previous chapter, this method was developed to solve the 4CP. For this reason, before talking about this method and giving the examples, we begin with a short story of the journey of the 4CP and the efforts that lead to the development of the Discharging Method.

2.1 The Four Colour Problem

This problem seems to have been first posed by Guthrie in 1852, when he was a law student at University College of London. He formulated this problem as a conjecture [35]:

> *"... the greatest necessary number of colours to be used in colouring a map so as to avoid identity of colours in lineally contiguous districts is four."*

In other words, we can colour any map of countries with four colours in such a way that any two countries sharing a common boundary segment (and not just a point) get different colours. When he could not solve the problem himself, Guthrie talked about this problem to his brother, who then passed it on to De Morgan. De Morgan couldn't come up with an answer either and gave the problem to Hamilton, but the problem did not draw his attention. In a note to De Morgan, Hamilton wrote: *"I am not going to attempt your quaternion of colour very soon"*. The first printed reference of the problem is due to Cayley in 1879 [22], in an article titled "On the colouring of maps". In this paper, Cayley explains to some extend, why this is a difficult problem. Before that, in 1860, Peirce too had attempted to solve this problem and didn't succeed.

This mysterious problem seemed to be solved in 1879 , when Kempe published the first "proof" of the 4CP in the American Journal of Mathematics [40]. Unfortunately, his proof was flawed, and surprisingly, it took mathematicians eleven years to notice the error, which was finally spotted by Heawood [36]. Another proof was proposed by Tait [53] in 1880. His proof was based on the assumption that every 3-connected 3-regular planar graph is Hamiltonian, which is not true. This gap was pointed out by Peterson in 1891, and the first explicit counter-example was found by Tutte [55] in 1946. However, both of these failed proofs were very useful; Heawood used a technique from Kempe's proof, which today is known as "Kempe chains", to prove that every map is five-colourable, and Tait found an equivalent formulation of the 4CP in terms of 3-edge-colouring.

The next major contribution came from Birkhoff [8] in 1913 who introduced the notion of *reducibility.* In a paper titled *"The reducibility of Maps"* he talked about configurations (sets of vertices and edges) that cannot exist in a minimum planar graph which cannot be 4-coloured. That is, a configuration that cannot be contained in a minimum counter-example to the 4CP. Franklin used this notion and went on to prove in 1922 that every planar map with at most 25 regions is four-colourable. This method was used by Reynolds in 1926 to prove the same statement for maps with up to 27 regions, then by Winn in 1940 for maps with 35 regions, by Ore and Stemple in 1970 for maps with 39 regions, and Mayer in 1976 for maps with 95 regions. However, this technique alone didn't seem to be sufficient to solve the 4CP for general planar graphs.

Heesch, in 1969, came up with a new idea, the method of *Discharging*, which later, together with the notion of reducibility, became the main ingredients used to solve the 4CP. Although he couldn't solve the 4CP himself, he conjectured that using the Discharging Method and considering 8900 reducible configurations, one can finish the job. The crucial rule of the Discharging Method was to prove the "unavoidability" of the set of reducible configurations. In other words, to prove that in any planar graph, one of these reducible configurations must exist, and therefore from the definition of a reducible configuration, there is no minimum counter-example to the 4CP.

In 1976, Appel and Haken [5] announced their proof of the Four Colour Theorem (4CT), in which they used the notion of reducibility and the Discharging Method. This proof used an extensive amount of computer time for verifying that more than 1400 configurations were reducible. They also had more than 300 discharging rules in the second step of their proof, which again used a computer to check all the possible cases. Overall, their proof needed more than 1200 hours of CPU time and it was inconceivable to manually check all the details of the proof.

This was the beginning of a controversy among mathematicians; should we consider such a proof as a "mathematical" proof? This is not an easy question, and mathematicians are still quite divided on its answer. The other, perhaps more serious, problem with the proof of the 4CT in particular, was that even those parts of the proof that were not automated and were supposed to be hand-checkable, were extremely complicated and nobody could verify them.

In 1996, Robertson, Sanders, Seymour, and Thomas [47] came up with yet another computer-aided proof of the 4CT. This proof is easier in that it has only 633 reducible configurations (compared to more than 1400 in the original proof by Appel and Haken [5, 7, 6]) and only 32 discharging rules. In explaining why they regenerated another proof of this theorem, Robertson et al. [47] list the following as the main two reasons the proof of Appel and Haken was not fully accepted:

(i) part of their proof uses a computer and cannot be verified by hand, and

(ii) even the part that is supposed to be checked by hand is extraordinarily complicated and tedious, and as far as we know, no one has made a complete and independent check of it.

However, reason (i) is an evil that still remains in the new proof, as pointed out by the authors. To verify this new proof and in particular part (i), an independent set of programs has been written by Fijavz under the guidance of Mohar (see the 4CT webpage at `http://www.math.gatech.edu/~thomas/FC/fourcolour.html`).

But some mathematicians still look at these proofs with skepticism. Thomas says: *"It is amazing that such a simply stated result resisted a proof for one and a quarter centuries, and even today it is not yet fully understood"*. Even today, some mathematicians are not satisfied with the proofs of the 4CT because they think such a nice and easy to explain problem must have a better and more understandable proof. Certainly, this proof is not from the "*book*"[1]. For more information on the 4CP see the nice survey by Claude [24].

While the most noteworthy application of the Discharging Method has been in the proof of the 4CT, there are dozens of other problems that have been solved using this technique. Some of the proofs are computer-aided, but the vast majority of them are hand-checkable. See, for example, [12, 13, 19, 20, 21, 51, 50]. Therefore, this method can be a handy tool for everybody who works on problems on planar graphs, and in many cases, on graphs embeddable on other surfaces, such as the projective plane and the torus.

[1] Very nice and elegant proofs are sometimes called "from the book" to refer to the *total book*, that Erdös believed might exist, and contains the best answers to every question.

2.2 How Does the Discharging Method Work?

Let Π be a class of planar graphs and suppose we want to prove that every graph in Π has a specific property P. We take an arbitrary graph $G \in \Pi$ and assign some charges to the elements of G (e.g. to the vertices, edges, or faces). Using Euler's formula, $|V|+|F|-|E| = 2$, we show that the total charge is some constant. Then we redistribute the charges according to some set of *discharging* rules that we define, while *preserving* the total charge. After this discharging phase, we show that either the total charge is now different (which of course is impossible) or G has some specific structures that imply property P. This technique is called the Discharging Method. Sometimes this method can be applied to problems for graphs embeddable on other surfaces, such as the projective plane or the torus, as Euler's formula holds for them with non-negative constants (1 and 0, respectively).

Often, we prove that the specific structures imply property P before applying the Discharging Method. The most common way to do this is to start the proof by way of contradiction and assume that there are graphs in Π that do not satisfy property P. Among all such graphs we consider one, called G_0, which has the smallest size. Then based on the assumption that G_0 is a minimum counter-example we prove that certain structures of vertices, edges, or faces cannot exist in G_0. These structures are called *reducible configurations*. Once a set of reducible configurations has been defined, we show that they are *unavoidable*. In other words, we prove that any graph in Π must have at least one of them. This proves that there is no minimum counter-example to the statement, or equivalently, every graph in Π has property P.

To do this second step, i.e. to prove unavoidability of the reducible configurations, we use the Discharging Method. That is, we take an arbitrary graph $G \in \Pi$ and apply the initial charges to G. Using Euler's formula we show that the total charge is, for instance, some negative constant. Then we apply the discharging rules and prove that either every element of G has non-negative charge (and so the total charge is non-negative), or G

must have one of these reducible configurations. Of course, the total charge must remain negative, since the discharging rules preserve the total charge. Therefore, there are some elements with negative charge in G. We prove that such elements must be in or near a reducible configuration.

Sometimes (as you will see soon) we don't use any set of reducible configurations. Instead, by applying a set of initial charges and the discharging rules, we can derive the required conclusion. However, in most applications of the Discharging Method, before applying the initial charges and the discharging rules, we come up with a suitable set of reducible configurations. For this reason, it is common to refer to both of the general steps explained above, i.e. the processes of finding a set of reducible configurations and proving the unavoidability of them, as the Discharging Method.

Here we demonstrate the use of this technique in a few examples. The first example is a well-know fact whose standard proof does not require the Discharging Method. We frame it in terms of the Discharging Method here for illustration of the technique, only.

Example 2.2.1 *Every simple planar graph $G(V, E)$ has a vertex of degree at most 5.*

Proof: Let F be the set of faces of G. To every vertex $v \in V$ with degree $d(v)$, we assign $d(v) - 6$ units of charge, and to each face $f \in F$ with size $|f|$ we assign $2|f| - 6$ units of charge. By noting that $2|E| = \sum_{v \in V} d(v) = \sum_{f \in F} |f|$, the total charge is: $\sum_{v \in V}(d(v) - 6) + \sum_{f \in F}(2|f| - 6) = 2|E| - 6|V| + 4|E| - 6|F| = 6(|E| - |V| - |F|) = -12$. Since the graph is simple, every face has size at least 3. So there must be a vertex with negative charge. Therefore, for some vertex v: $d(v) - 6 < 0$, that is $d(v) \leq 5$, as wanted. ■

The above example was easy and we did not have to move any of the charges. The next one is less trivial and contains some charge movement; i.e., a discharging phase.

Example 2.2.2 *In every simple planar graph $G(V, E)$ with minimum degree at least three, there is a vertex of degree d incident with a face of length l such that $d + l \leq 8$.*

Proof: We call a vertex-face incidence a *corner*. To every vertex $v \in V$ with degree $d(v)$ we assign a charge of $d(v)-4$, and to each face $f \in F$ with length $|f|$ we assign a charge of $|f|-4$. Again, using Euler's formula, the total charge is: $\sum_{v\in V}(d(v)-4)+\sum_{f\in F}(|f|-4) = 2|E|-4|V|+2|E|-4|F| = 4(|E|-|V|-|F|) = -8$. In the discharging phase every vertex v sends out $\frac{d(v)-4}{d(v)}$ units of charge to each corner that it participates in. Similarly, each face f sends $\frac{|f|-4}{|f|}$ charge to each corner that it belongs to. Therefore, after the discharging phase, all the vertices and faces have charge 0. Since the total charge was negative, there must be a corner with negative charge. Assume that this corner is made from the incidence of a vertex v with $d(v) = d$ and a face f with $|f| = l$. The charge of this corner is $\frac{d-4}{d} + \frac{l-4}{l} < 0$. Therefore $2ld - 4l - 4d < 0$, which together with the assumptions that the minimum degree is at least three and each face has size at least three, imply:

$$d < \frac{2l}{l-2} \leq 6 \quad \text{and} \quad l < \frac{2d}{d-2} \leq 6.$$

Adding l to both sides of the first inequality yields $d+l < \frac{l^2}{l-2}$, which is at most 8 for $3 \leq l < 6$. ∎

The next example is more involved. It is actually a simplified version of the problem that is considered in Chapters 3 and 4. We will talk about the history of this problem and the previous results on this in more detail in Chapter 3.

Example 2.2.3 *(Abbott and Zhou [1]) Every planar graph without any cycle of size in* $\{4, \ldots, 11\}$ *is 3-colourable.*

Proof: The proof contains two main parts:

Part 1 (Reducible Configurations): By contradiction, assume that the statement is false and let $G(V, E)$ be a counter-example with the minimum number of vertices. So G is a 4-critical graph. Trivially, G must be connected. We claim that (i) a vertex with degree at most 2, and (ii) a cut-vertex are reducible configurations.

(i) Suppose that $v \in V$ is a vertex with degree at most 2. Because G is 4-critical there exists a 3-colouring of $G - \{v\}$. As v is adjacent to at most 2 vertices, we can extend this colouring to v by assigning a colour different from its neighbours and obtain a 3-colouring of G, a contradiction.

(ii) Suppose that $v \in V$ is a cut-vertex and $C_1, \ldots, C_k$ are the connected components of $G - \{v\}$, with $k \geq 2$. By definition of G, each $C'_i = C_i \cup \{v\}$, $1 \leq i \leq k$, has a 3-colouring $\phi_i : V(C'_i) \longrightarrow \{1, 2, 3\}$. Now, without loss of generality, and by possibly permuting the colours in some of ϕ_i's, we can assume that $\phi_i(v) = 1$, for $1 \leq i \leq k$. The union of these colourings gives a 3-colouring of G, a contradiction.

Part 2 (Discharging): Now we prove that this set of reducible configurations is unavoidable, i.e. any planar graph without cycles of size in $\{4, \ldots, 11\}$ has at least one of them. This shows that there is no minimum counter-example (and therefore no counter-example at all) to the statement. Let G be any planar graph without any cycle of size in $\{4, \ldots, 11\}$. To each vertex $v \in V$ with degree $d(v)$ we assign a charge of $d(v) - 6$, and to each face f with size $|f|$ we assign $2|f| - 6$. The total charge is: $\sum_{v \in V}(d(v) - 6) + \sum_{f \in F}(2|f| - 6) = 2|E| - 6|V| + 4|E| - 6|F| = -12$. Since each face has size at least 3, all faces have non-negative charge. If G has a vertex of degree at most 2, since it is one of the reducible configurations described in Part 1, we are done. Otherwise, the minimum degree of G is at least three, and therefore, the only vertices with negative charge are vertices with degree 3, 4, or 5.

In the discharging phase, every face f with $|f| \geq 12$ sends $\frac{3}{2}$ units of charge to each of its vertices. An important observation to make here is that since G does not have any cycle of size 4, it cannot have two faces f_1, f_2, each of size 3, that have an edge in common. If G has a cut-vertex then we are done, since that is a reducible configuration. Otherwise, every vertex $v \in V$ is incident with at least $\lceil \frac{d(v)}{2} \rceil$ *distinct* faces that have size at least 12, each. Consider an arbitrary vertex v:

- If $3 \leq d(v) \leq 5$ then it gets a total of at least $\frac{3}{2} \times \lceil \frac{3}{2} \rceil = 3$. Its initial charge was at least $d(v) - 6 \geq -3$, and therefore, it has non-negative charge.

- if $d(v) \geq 6$, it had originally non-negative charge and it does not lose any charge in the discharging phase.

So all the vertices have non-negative charge. Faces of size 3 had originally a charge of 0 and they don't lose any charge in the discharging phase. There are no cycles of size in $\{4, \ldots, 11\}$, and therefore no faces of size in $\{4, \ldots, 11\}$. Every other face f has size at least 12 and it sends out $\frac{3}{2}|f|$ units of charge which is not more than $2|f| - 6$, for $|f| \geq 12$. Thus, all faces have non-negative charge after the discharging phase. However, the total initial charge was -12. This contradiction completes the proof. ■

Now, one might ask if we can improve this statement by allowing cycles of size 11. In other words, can we still prove 3-colourability if the given planar graph does not have cycles of size in $\{4, \ldots, 10\}$? You will see in a moment that by being a little bit more careful in the design of the discharging rules we can prove this, using the same set of reducible configurations.

Example 2.2.4 *(Borodin [15]) Every planar graph without any cycle of size in* $\{4, \ldots, 10\}$ *is 3-colourable.*

Proof: Part 1 (Reducible Configurations): It is easy to see that the two reducible configurations in the previous proof, i.e. a vertex with degree at most 2 (a ≤ 2-vertex) and a cut-vertex, still form a set of reducible configurations.

Part 2 (Discharging): Let G be any planar graph without cycles of size in $\{4, \ldots, 10\}$ and apply the same set of initial charges to G. That is, to each vertex v we assign $d(v) - 6$ and to each face f we assign $2|f| - 6$ units of charge. Recall that by Euler's formula the total charge is -12. Now we have to define the set of discharging rules and show that after the discharging phase either we have one of the reducible configurations, or the total charge is non-negative, which of course is impossible. If we use the same set of discharging rules as in the previous proof everything works out up to faces of size 11,

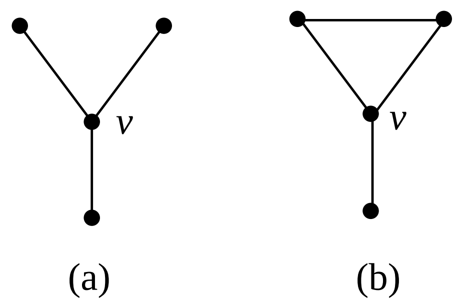


Figure 2.1: (a) A simple vertex and (b) a bad vertex

i.e., we can show that either we have a reducible configuration (a ≤ 2-vertex or a cut-vertex) or all the vertices and all the faces of size at least 12 have non-negative charge. To complete the proof we need to show that none of the faces of size 11 will end up with negative charge, either. But this is not true, because each such face sends out $\frac{3}{2} \times 11$, which is larger than its initial charge 16. But, do we really need to send $\frac{3}{2}$ from each face to all the vertices incident with it?

It is not hard to see that 3-vertices (with initial charge of -3) are the most desperate vertices for charge. If a 3-vertex v is incident with exactly one triangular face we call it a *bad vertex* and a 3-vertex which is incident to no triangular face is called *simple* (see Figure 2.1). Note that by absence of 4-cycles, every 3-vertex is either simple or bad.

Since triangular faces have charge 0, they cannot afford to send any charges out in the discharging phase. Therefore, if v is a bad vertex then each of the two non-triangular faces that v is incident with, must send $\frac{3}{2}$ to v. So every face f must send $\frac{3}{2}$ to each of its bad vertices. But if v is a simple vertex, then it is incident with three non-triangular faces, and therefore, can receive charges from each of them. So it will be sufficient to send only 1 unit of charge from each of those faces to v. Also, if v is a ≥ 4-vertex, its initial charge (which is $d(v) - 6$) is at least -2 and, as in the proof of Example 2.2.3, it is incident with at least two non-triangular faces. If each of those faces sends 1 unit of charge to v then v will have non-negative charge. This way, we may save enough charge on faces, so much so that faces of size 11 have non-negative charge, too. So let's modify the discharging rule to:

> Every non-triangle face f sends $\frac{3}{2}$ units of charge to each of its bad vertices and 1 unit of charge to each of its other vertices.

As before, if we have a ≤ 2-vertex or a cut-vertex we are done. Otherwise, by this discharging rule every 3-vertex v receives at least 3 units of charge: if v is a bad vertex it receives $\frac{3}{2}$ units from each of the non-triangular faces it is incident with, and if it is a simple vertex it receives 1 unit of charge from each of the three faces it is incident with. Also, as we proved above, every ≥ 4-vertex receives at least 2 units of charge and will have non-negative charge. Regarding the faces, they are not sending more charges than in the previous example, and therefore, faces of size at least 12 have non-negative charge by the proof of Example 2.2.3. For a face f of size 11, an important observation to make is that it can be incident with at most 10 bad vertices, because of parity (bad vertices on a face come in pairs). Therefore, f sends out at most $10 \times \frac{3}{2} + 1 = 16 = 2|f| - 6$, and hence has non-negative charge, which completes the proof.

■

If we want to relax the condition further and allow cycles of size 10 then this set of discharging rules does not seem to work, since a face of size 10 may be incident with 10 bad vertices, and therefore, must send $10 \times \frac{3}{2} > 2|f| - 6$. So, one might think that to improve the result of Example 2.2.4 one step further, we should try to come up with a better set of discharging rules, and possibly a more careful assignment of the initial charges. Figure 2.2 shows a planar graph which does not have any cycle of size in $\{4, \ldots, 9\}$ and, neither has a ≤ 2-vertex nor a cut-vertex. Therefore, changing only the discharging part of the proof does not help and we must look for a new reducible configuration.

In general, to improve a result that uses the Discharging Method, sometimes it is enough find a better set of initial charges and discharging rules (as we did in Example 2.2.4). But there are some situations (as described in the previous paragraph) that there

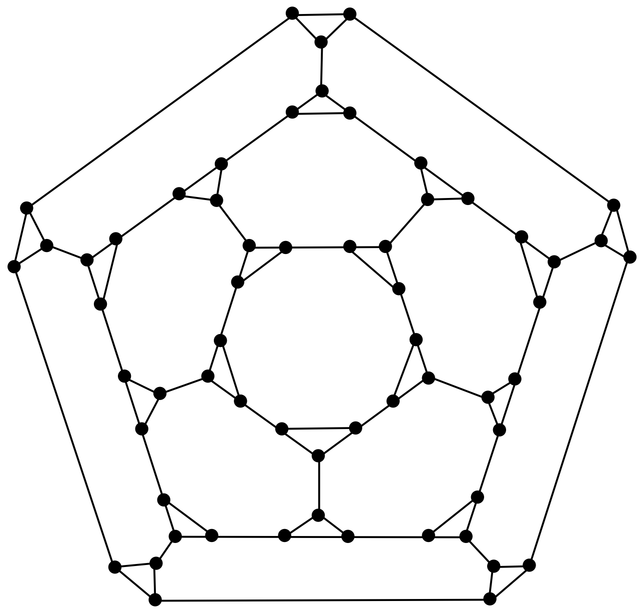


Figure 2.2: A 2-connected planar graph with $\delta \geq 3$ and without cycles of size in $\{4, \ldots, 9\}$

is no way of improving a result just by changing the discharging part since there are graphs that do not have any of the current reducible configurations. In those situations we must find a new set of reducible configurations and possibly a new set of discharging rules that work with them. Almost always these two processes (finding a set of reducible configurations and designing a set of discharging rules that work with them) have to be co-ordinated. That is, looking at a current set of discharging rules gives us some hints as to what kind of new reducible configurations we should be looking for.

One way of doing this is by looking at the elements that have negative charge after applying the current set of discharging rules, but do not lie in or near a member of our current set of reducible configurations. (Of course, we must have such an element, or else we would already have a complete proof.) Often these elements are in or near something that we can prove to be a new reducible configuration. For instance, when we tried to apply our previous set of rules to graphs with 10-cycles, we saw that a 10-face with 10 bad vertices got negative charge. This inspires us to show, in the next chapter, that such a face is reducible. (Note that the graph in Figure 2.2 has many such faces.) On the other hand, sometimes we cannot find new reducible configurations. Then we should refine our discharging rules to send more charge to those problematic negatively charged elements. Very roughly speaking, one can say that the relation of processes of finding these two sets (the reducible configurations and the discharging rules) is similar to the relation between the primal and dual of a linear program in the design of algorithms based on a primal-dual scheme. Hopefully these rough statements will be clearer in Chapter 3, when we explain how to improve the last example.

One final point worth mentioning is that in proofs using the Discharging Method, there are usually equivalent forms of assigning initial charges to the elements of the graph, in the sense that the proof based on one set of charges can be translated into a proof based on another one, using a linear transformation of the initial charges and the discharging rules. Furthermore, to be able to use Euler's formula to calculate the sum of the initial charges there are a limited number of forms of initial charges we can use. So, the role in the proof that the set of initial charges play is not as crucial as that played by the discharging rules.

2.3 Designing Algorithms Using the Discharging Method

We close this chapter by noting that almost all proofs using the Discharging Method are constructive and yield efficient polynomial time algorithms. Usually, the reducible configurations have size bounded by a constant k and so, naively, we can just do exhaustive search and find one in time $O(n^k)$. Often, the Discharging Method helps to do this step faster. For instance, the proof of Example 2.2.4 yields an $O(n^2)$ time recursive algorithm such that for a given embedded planar graph G without cycles of size in $\{4, \ldots, 10\}$ produces a 3-colouring of G. For a disconnected graph, trivially it is enough to colour each connected component separately. Therefore we give a procedure for 3-colouring connected planar graphs without cycles of size in $\{4, \ldots, 10\}$.

Each iteration of the procedure consists of either finding a ≤ 2-vertex and removing it to obtain a smaller graph, or finding a cut-vertex and breaking the graph into smaller subgraphs. Then we colour the new smaller graph(s) recursively, and extend the colouring(s) to the whole graph. We keep doing this as long as there is at least one vertex in the given graph. Here are the steps of the procedure:

- Apply the initial charges and the discharging rule.
- Since the total charge is negative, there is some vertex with negative charge (note that by the proof of Example 2.2.4 all faces will have non-negative charge).
- If $v \in V$ has negative charge, then either $d(v) \leq 2$ or v is a cut-vertex. We can check whether $d(v) \leq 2$ or not in constant time. If $d(v) \leq 2$ then we find a 3-colouring for each connected component of $G - v$, recursively. These colourings can be easily extended to G, since v has at most two coloured neighbours.

  If v is a cut-vertex and the connected components of $G - v$ are $C_1, \ldots, C_k$, then we find a 3-colouring for each $G_i = C_i \cup \{v\}$, recursively. The union of these colourings, possibly permuting the colours in each, yields a 3-colouring of G.

Now we analyze the running time of this procedure. Since in a planar graph the number of edges and faces is linear in the number of vertices, we consider the size of a planar graph to be the number of vertices in it. Let $T(n)$ be the worst case running time of the procedure on an input graph of size n. In each iteration we apply the initial charges and the discharging rule. For each face f it takes $O(|f|)$ time to apply the discharging rule to it. Since only faces send charges in the discharging phase, this step takes at most $O(\sum_{f \in F} |f|)$ time which is in $O(n)$. Then we find a vertex with negative charge which can be done in $O(n)$ time. Extending the colourings of the smaller subgraphs (that are obtained recursively) to G takes constant time for the case that the vertex with negative charge was a ≤ 2-vertex, and takes at most $O(n)$ time for the case that it was a cut-vertex (since we may have to permute the colours in some of the subgraphs).

We prove by induction that for some constant $C > 0$ and all values of $n \geq 1$: $T(n) \leq Cn^2$. The inequality is trivial for small values of n. So let's assume that for all values of $1 \leq i < n$: $T(i) \leq Ci^2$. Consider the procedure call in which the input graph has size n. If a $\leq$2-vertex is found we make recursive calls on at most two smaller graphs of sizes n_1 and n_2, respectively, with $n_1+n_2 = n-1$. Therefore: $T(n) \leq \alpha n+T(n_1)+T(n_2)$, for some constant $\alpha > 0$. Thus:

$$\begin{aligned} T(n) &\leq \alpha n + T(n_1) + T(n_2) \leq \alpha n + Cn_1^2 + Cn_2^2 \\ &\leq \alpha n + C(n_1 + n_2)^2 \leq Cn^2, \end{aligned}$$

where the last inequality holds if C is large enough with respect to α.

If a cut-vertex is found we make recursive calls on k smaller graphs $G_1, \ldots, G_k$, with $2 \leq k \leq n-1$. Let $n_i = |V(G_i)|$, $1 \leq i \leq k$. Note that $2 \leq n_i \leq n-1$ (for $1 \leq i \leq k$) and $\sum_{i=1}^{k}(n_i - 1) = n - 1$. Therefore, for some constant $\alpha > 0$:

$$T(n) \leq \alpha n + \sum_{i=1}^{k} T(n_i) \leq \alpha n + C \sum_{i=1}^{k} n_i^2.$$

The last summation is is maximized when $k = 2$ and one of n_1 or n_2 is $n-1$ and the other is 2. At this maximum, the sum is easily seen to be less than Cn^2, as wanted.

2.3.1 An Extended Algorithm for Example 2.2.4

In Chapter 4 we will need to use a stronger version of the algorithm given above. Here we describe this new algorithm. The input to this algorithm is an embedded planar graph G without cycles of size in $\{4, \ldots, 7\}$. The algorithm either comes up with a 3-colouring of G or finds a cycle of size in $\{4, \ldots, 10\}$. Again, we assume that the input graph is connected, as for disconnected graphs it is enough to run the algorithm on each connected component, independently.

At each iteration of the algorithm, we apply the initial charges and then the discharging rule as described in the proof of Example 2.2.4. Since the total charge is negative there must be some element with negative charge after the discharging phase. If there is no face of size in $\{8, 9, 10\}$, then the only elements with negative charge will be 2-vertices and cut-vertices. If we find such a vertex with negative charge we proceed as in the previous algorithm. The other possibility is to have a face f with negative charge. Such a face must be a face with size in $\{8, 9, 10\}$. Since the input at each iteration of the algorithm is a subgraph of the original graph, this face f that has negative charge corresponds to a cycle of G. Therefore, at each iteration of the new algorithm, either we find a cycle of size in $\{8, 9, 10\}$ and the algorithm terminates and returns it, or we find a ≤ 2-vertex or a cut-vertex and we proceed as in the previous algorithm. It is easy to see that this slight modification does not change the running time of the algorithm, and therefore, this new algorithm runs in $O(n^2)$ time, too.

Chapter 3

The Three Colour Problem

Remark 3.0.1 *The results in this chapter are based on paper [48].*

3.1 Steinberg's Conjecture

In 1959, almost two decades before the Four Colour Theorem was proved, Grötsch [33] showed that every planar graph without 3-cycles is 3-colourable. In 1976, Steinberg [4, 52] conjectured that every planar graph without 4- and 5-cycles is 3-colourable. Both 4- and 5-cycles must be excluded. In fact there is an infinite family of 4-critical planar graphs that have only four 4-cycles and no 5-cycles, and there is an infinite family of 4-critical planar graphs that have no 4-cycles and have only six 5-cycles [1]. An equivalent formulation of this conjecture is that every 4-chromatic planar graph has a 4- or 5-cycle. This problem is also discussed in the monograph by Jensen and Toft [38] (Problem 2.9).

In 1990, Erdös relaxed the conjecture of Steinberg by asking if there exists an integer $k \geq 5$, such that every planar graph without cycles of size in $\{4, \ldots, k\}$ is 3-colourable. An affirmative answer to the question of Erdös (and therefore a partial answer to the conjecture of Steinberg) was obtained by Abbott and Zhou [1], who showed that $k = 11$ is suitable, i.e. any planar graph without cycles of size in $\{4, \ldots, 11\}$ is 3-colourable. This is in fact our Example 2.2.3 in Chapter 2. Borodin [15] improved this result to

$k = 10$ (Example 2.2.4). To the date we started working on the problem, the best known answer, which states that $k = 9$ is suitable, was due to Borodin [14] and independently to Sanders and Zhao [49] (An erroneous proof for the case $k = 8$ was claimed by B. Xu [61], but it was later withdrawn).

Let $\mathcal{G}_8$ be the class of planar graphs without cycles of size in $\{4, \ldots, 8\}$. The main result of this chapter is:

Theorem 3.1.1 *Every graph in $\mathcal{G}_8$ is 3-colourable.*

The proof of this theorem is constructive and yields an $O(n^2)$ time algorithm for finding a 3-colouring of such graphs.

One key idea in the proof of this theorem, that distinguishes it from the previously known results, is the following. To prove the reducibility of (some of) the configurations, we modify the configuration by removing some vertices and edges and by adding a smaller number of vertices and edges, which will be called the "gadget". This modification is designed carefully so that it enforces some properties that we require to prove the reducibility, while preserving planarity and the key property of not having any cycles of size in $\{4, \ldots, 8\}$. Therefore, the new graph is in $\mathcal{G}_8$, and since the graph we started with was a minimum counter-example, there must be a 3-colouring of this new graph. Then using the properties of the gadget we have added, we show how this 3-colouring can be extended to a 3-colouring of the original graph.

The total number of reducible configurations used in the proof of this theorem is much larger than in the previous results; it is about[1] 77, compared to 3 configurations needed to prove the previous best known bound. To simplify the presentation of the proof, we have divided these configurations into several groups based on their structures. We have checked the reducibility of these 77 configurations by hand, but writing a hand-checkable proof for each configuration and also going through these proofs and checking every single

[1] The actual number of reducible configurations is 69, since reducibility of some of them follows from the others. However, the presentation of the proof will be significantly easier if based on 77 configurations.

configuration by hand is a lengthy and tedious task. Instead, we give the hand-checkable proofs of some of the configurations in each group. These proofs have a very similar structure and, after seeing a few of them, checking the reducibility of the other ones becomes straightforward (although tedious and time consuming). But that's not all! We have verified the reducibility of all of these configurations using a short and simple C program. So, we also have a computer-aided proof. The program and the list of all the reducible configurations appear in Appendices B and C, where we also explain how the configurations have been generated. In Section 3.5 we explain how this program works. The program and the file containing the reducible configurations and the description of the program is also available at `ftp://ftp.cs.toronto.edu/csrg-technical-reports/458/`.

The organization of this chapter is as follows. Instead of proving Theorem 3.1.1 right away, in the next section we first try to improve on Example 2.2.4. We do this by looking back at the proof of that example. This will lead us to prove a weaker version of Theorem 3.1.1, which is basically the result of Borodin [14] and Sanders and Zhao [49]. The proof of Theorem 3.1.1 is provided in Section 3.3. We present some more notation and definitions in Subsection 3.3.1. Subsection 3.3.2 contains the description of all the reducible configurations and the hand-checkable proofs of some of them. We explain the discharging rules in Subsection 3.3.3, which also completes the proof of Theorem 3.1.1. Appendix A contains more hand-checkable proofs of reducible configurations. In Section 3.4 we show how the proof of Theorem 3.1.1 yields a quadratic time algorithm for 3-colouring of graphs in $\mathcal{G}_8$. Finally, in Section 3.5 we talk about the automated proof of reducibility and the program written for this purpose.

3.2 A Weaker Version of the Main Theorem

Let $\mathcal{G}_9$ be the class of planar graphs without any cycle of size in $\{4, \ldots, 9\}$. Our goal is to prove:

Theorem 3.2.1 *Every graph in $\mathcal{G}_9$ is 3-colourable.*

This is the previously best known result on this problem, proved by Borodin [14] and by Sanders and Zhao [49]. To prove Theorem 3.2.1, we look back at the proofs of Examples 2.2.3 and 2.2.4, and try to find a weakness in the arguments and improve it.

Recall the proof of Example 2.2.4. We showed that a minimum counter-example cannot have a cut-vertex or a ≤ 2-vertex, i.e. these two are reducible configurations. Then to show that any arbitrary planar graph G without cycles of size in $\{4, \ldots, 10\}$ has one of these two reducible configurations we assigned $d(v) - 6$ units of charge to every vertex v and $2|f| - 6$ units to every face f of G. In the discharging phase, every non-triangle face f sent $\frac{3}{2}$ to each of its bad vertices and 1 to each of its other vertices.

This argument fails to work for graphs in $\mathcal{G}_9$ since for faces of size 10, the total charge sent out might be more than their initial charges, and therefore, a 10-face may have negative charge after the discharging phase. That happens, for example, to every non-triangle face of the graph in Figure 2.2. The main problem here is that this graph does not have any of the two reducible configurations (a ≤ 2-vertex and a cut-vertex). Note that every 10-face in this graph has 10 bad vertices. This inspires us to try to prove that if f is a 10-face in a minimum counter-example to Theorem 3.2.1, then there are some non-bad vertices in the boundary of f. At least two non-bad vertices will be enough since then in the discharging part of our proof, a non-reducible face f of size 10 would send out at most $8 \times \frac{3}{2} = 12$ (to the bad vertices) and $2 \times 1 = 2$ (to the non-bad vertices) for a total of $14 = 2|f| - 6$, and so would have non-negative charge after the discharging phase.

Note that every minimum counter-example to Theorem 3.2.1 is 4-critical as if $G \in \mathcal{G}_9$ then $G - e \in \mathcal{G}_9$ for every $e \in E(G)$.

Claim 3.2.2 *No 4-critical planar graph (and therefore no minimum counter-example to Theorem 3.2.1) has a $2k$-face f $(k \geq 2)$ with at least $2k - 1$ bad vertices.*

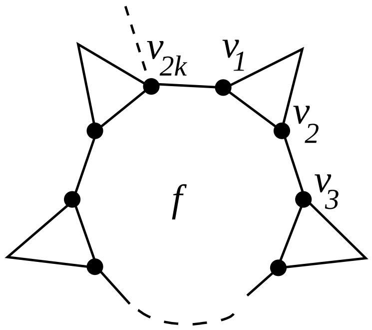


Figure 3.1: A $2k$-face incident with at least $2k-1$ bad vertices

Proof: Let G be a 4-critical planar graph and let f be a $2k$-face of G whose vertices in clockwise order are $v_1, \dots, v_{2k}$. By way of contradiction assume that $v_1, \dots, v_{2k-1}$ are all bad vertices. This implies that each has degree 3 and is incident with exactly one triangle. Without loss of generality, we can assume that v_{2i-1} and v_{2i} are incident with the same triangle, $1 \leq i \leq k$. Thus, v_{2k} either is also bad or has degree at least four. (see Figure 3.1). Since G is 4-critical, there is a 3-colouring of $G - v_1v_{2k}$, called ϕ. Because G is not 3-colourable, $\phi(v_1) = \phi(v_{2k})$; without loss of generality, we can assume both are 1. We claim $\phi(v_3) = 1$, otherwise we could exchange $\phi(v_1)$ with $\phi(v_2)$ and obtain a 3-colouring of G. Using a similar argument, we can show that $\phi(v_5) = 1$, and in general by induction, one can easily prove that $\phi(v_{2i+1}) = 1$, for $0 \leq i \leq k-1$. But $\phi(v_{2k-1})$ cannot be equal to 1, as it is adjacent to v_{2k} and $\phi(v_{2k}) = 1$. This contradiction completes the proof. ■

Remark 3.2.3 *Note that the proof of this claim actually shows that any 3-colouring of $G - v_1v_{2k}$ can be extended to a 3-colouring of G in constant time (for constant k), by only exchanging the colours of some of the vertices of f.*

Now we have a new set of reducible configurations (the first two were proved to be reducible in Example 2.2.3 and those proofs clearly extend to this setting):

- a vertex of degree at most 2,

- a cut-vertex, and
- a $2k$-face with at least $2k-1$ bad vertices.

We will use the Discharging Method to prove that every planar graph $G \in \mathcal{G}_9$ must contain at least one of these configurations.

Proof of Theorem 3.2.1: The set of initial charges and the discharging rules are the same as in Example 2.2.4. Recall that by Euler's formula the total initial charge is -12. By the arguments of the proof of Example 2.2.4, either we have a ≤ 2-vertex or a cut-vertex, or every vertex and every face of size at least 11 has non-negative charge. If G has a ≤ 2-vertex or a cut-vertex we are done. Otherwise, because the total charge must remain negative after the discharging phase, there must be a face of size 10 with negative charge. Suppose f is such a face. As we discussed before Claim 3.2.2, f must be incident with at least 9 bad vertices to have negative charge. But such a structure is reducible by Claim 3.2.2. Therefore, G contains one of the reducible configurations and this completes the proof of Theorem 3.2.1. ■

Note that, as does the proof of Example 2.2.4, this proof yields a simple quadratic time 3-colouring algorithm. Here we give a procedure that given a connected embedded graph $G \in \mathcal{G}_9$ as input, produces a 3-colouring of G. Obviously if we have a disconnected graph $G \in \mathcal{G}_9$, it is enough to apply this procedure to each of its connected components. At each iteration of the procedure, we apply the initial charges and then the discharging rule. Since the total charge is negative, there is either a vertex or a face with negative charge:

1. If there is a vertex v with negative charge, then either v is a ≤ 2-vertex or a cut-vertex. As in the algorithm of Example 2.2.4, for the case that v is a ≤ 2-vertex we can colour each connected component of $G - \{v\}$, recursively, and extend these colourings to v. For the case that v is a cut-vertex and $C_1, \ldots, C_t$ ($t \geq 2$) are the connected components of $G - \{v\}$, we can colour each $G_i = C_i \cup \{v\}$, $1 \leq i \leq t$,

recursively, and then combine these colourings (possibly permuting some colours in some of the colourings) to obtain a 3-colouring of G.

2. If there is a face f with negative charge, then this face must be a 10-face with at least 9 bad vertices. We remove one of the edges as in the proof of Claim 3.2.2 and colour the new graph recursively. By Remark 3.2.3, this colouring can be extended to G in constant time.

This procedure iterates as long as there is at least one vertex in the graph. Let the size of the input graph G be $n = |V| + |E|$ and denote the worst case running time of the algorithm on an input of size n by $T(n)$. As in the algorithm of Section 2.3, since faces are the only elements that send charges in the discharging phase, applying the initial charges and the discharging rule takes $O(\sum_{f \in F} |f|)$ time, which is in $O(n)$. After that, finding an element (vertex or face) with negative charge takes $O(n)$ time. If the element with negative charge is a vertex then (as we had in the algorithm of Example 2.2.4) it takes at most linear time to extend the colouring of the smaller graphs to G. If the element with negative charge is a face, by Remark 3.2.3, it takes constant time to extend the colouring to G. So we can assume that all these steps take at most αn time, for some constant $\alpha > 0$.

By induction on n, we prove that for all values of $n \geq 1$ and for a suitable constant $C > 0$: $T(n) \leq Cn^2$. For small values of n the inequality trivially holds, if C is large enough. Suppose that $T(i) \leq Ci^2$ for all values of $1 \leq i < n$, and consider the iteration in which the input graph has size n. After the discharging phase:

- For the case that the element with negative charge is a 2-vertex or a cut-vertex then an analysis very similar to that of algorithm of Section 2.3 shows that $T(n) \leq Cn^2$.

- For the case that the element with negative charge is a face then we make a recursive call on a graph obtained by removing a single edge of G, i.e. a graph with size $n-1$. Therefore: $T(n) \leq \alpha n + T(n-1) \leq \alpha n + C(n-1)^2 \leq Cn^2$, for large enough C.

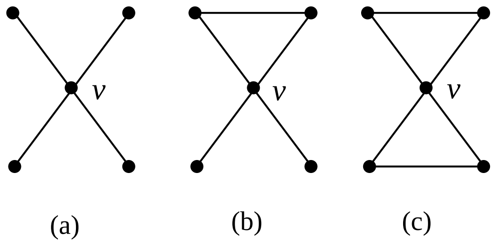


Figure 3.2: (a) A type 0 vertex, (b) a type 1 vertex, and (c) a type 2 vertex

So the running time of the algorithm is $O(n^2)$.

3.3 Proof of the Main Theorem

In this section, we strengthen our arguments from the previous section to prove Theorem 3.1.1. First we need to state a few more definitions used in the description of reducible configurations. We will also use the definitions from Chapter 2 for bad and simple vertices.

3.3.1 Preliminaries

Recall that a 3-vertex is bad if it is incident with exactly one triangle, and simple otherwise. Let v be a vertex with degree 4. Then v is called a *type 0* vertex if it is not incident with any triangles. If it is incident with exactly one or exactly two triangles, then it is called a *type 1* or *type 2* vertex, respectively. Note that by absence of 4-cycles, every 4-vertex is one of these three types (See Figure 3.2).

In the proof of Theorem 3.2.1 we saw how to deal with faces of size at least 10. Dealing with 9-faces will require some care. We begin by defining some structures that involve 9-faces. Let f be a 9-face incident with 8 bad vertices. Then f is called a simple, a type 0, a type 1, a type 2, or a type 5 face, if the ninth vertex of f is a simple, a type 0, a type 1, a type 2, or a 5-vertex, respectively (See Figure 3.3).

Now suppose that f is a 9-face which has exactly 7 bad vertices (and therefore is

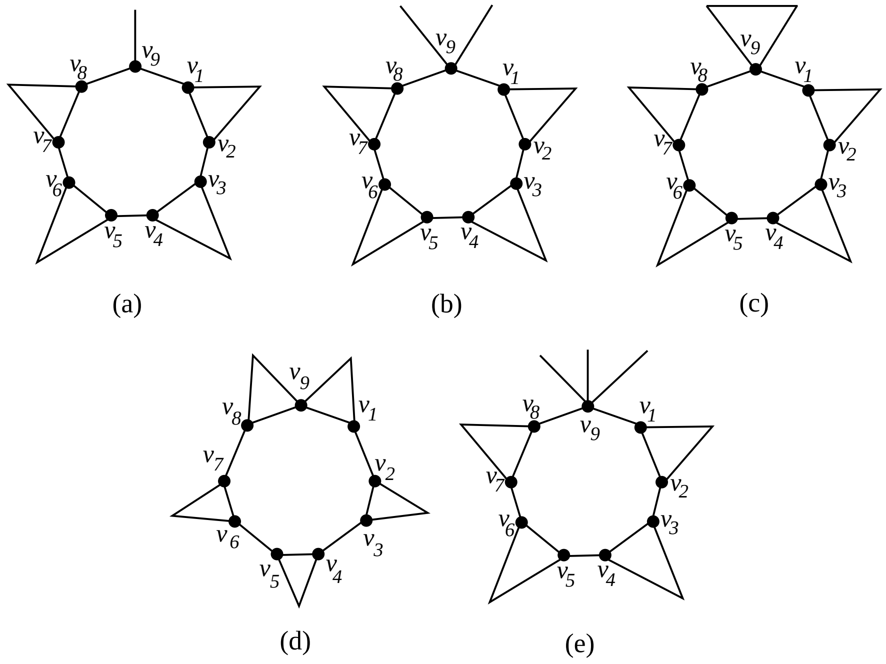


Figure 3.3: (a) A simple face, (b) a type 0 face, (c) a type 1 face, (d) a type 2 face, and (e) a type 5 face.

adjacent to exactly four triangles), and has a type 1 vertex which is incident with one of these four triangles. This accounts for 8 of the vertices. If the ninth vertex of f is a simple vertex then f is called a *semi-simple* face. Similarly, if the ninth vertex of f is a type 0 vertex, or a type 1 vertex, or a type 2 vertex, then it is called a *semi-type 0*, *semi-type 1*, or *semi-type 2* face, respectively (see Figure 3.4). We have not given a name to every 9-face. We named only 9-faces with 8 bad vertices and a ≤ 5-vertex, or with 7 bad vertices and two 4-vertices.

As you will see later, some reducible configurations are made from an interaction of three faces of size 9. For this reason we have to define a few more structures. Let f_1 be a semi-type 0 face whose vertices (in counter-clockwise order) are $v_1, v_2, \ldots, v_9$, where v_1 is the type 0 vertex, and let f_2 be a type 0 face whose type 0 vertex is v_1. If v_i is the type 1 vertex of f_1, for some $3 \leq i \leq 8$, and f_3 is a semi-simple face whose type 1 vertex is v_i, then we call this configuration a "*simple triple structure*" (See Figure 3.5(a)). Similarly,

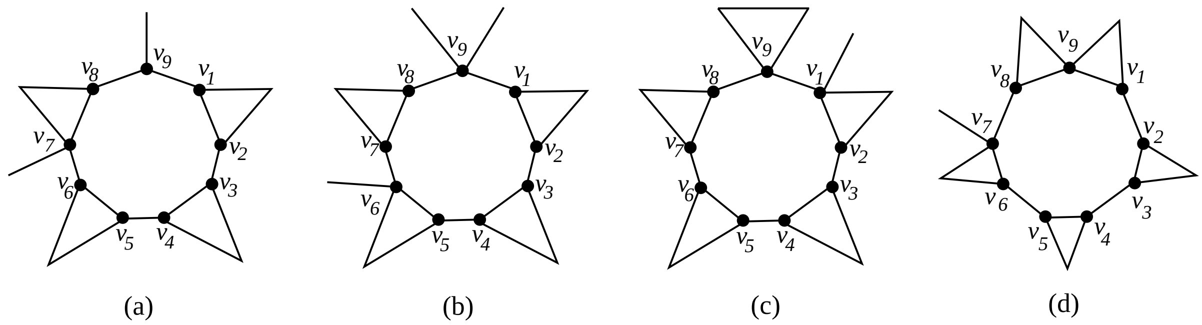


Figure 3.4: (a) A semi-simple face, (b) a semi-type 0 face, (c) a semi-type 1 face, (d) a semi-type 2 face

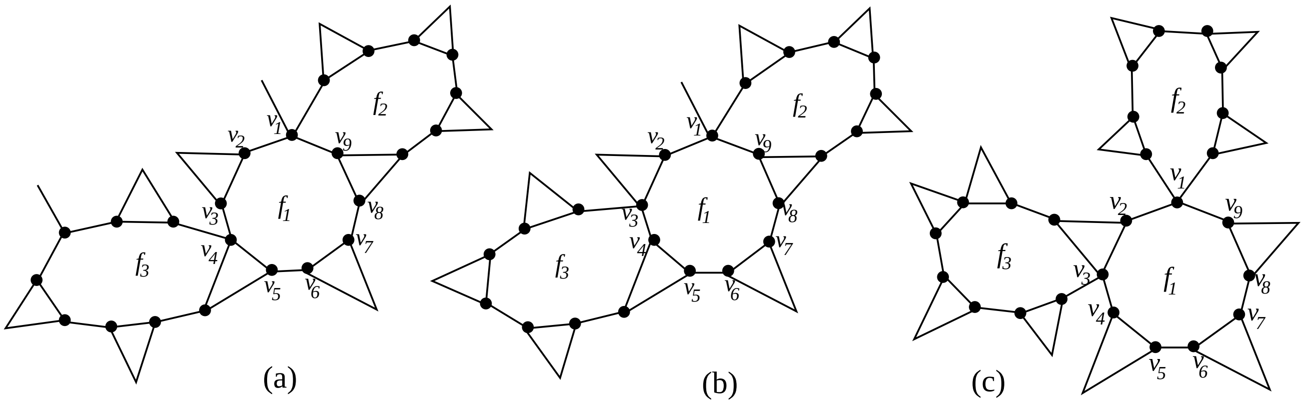


Figure 3.5: (a) A simple triple structure with $v_i = v_4$, (b) a triple structure of kind 1 with $v_i = v_3$, (c) a triple structure of kind 2 with $v_i = v_3$

if f_3 is a type 1 face whose type 1 vertex is v_i, then we call this configuration a "*triple structure of kind 1*". Finally, if f_3 is a semi-type 2 face whose type 1 vertex is v_i, then we call this configuration a "*triple structure of kind 2*". (See Figure 3.5)

3.3.2 Reducible Configurations

Suppose we were to follow the same steps as in the proof of Theorem 3.2.1. That is, in the discharging part assign an initial charge of $d(v) - 6$ to every vertex v and $2|f| - 6$ to every face f. For the moment, let's assume that we used the same discharging rule, i.e. every non-triangle face f sends $\frac{3}{2}$ to every bad vertex and 1 to every other vertex in its boundary. Then, as in the proof of Theorem 3.2.1, we could show that either we have one of the reducible configurations of the proof of Theorem 3.2.1, or every vertex and every face of size at least 10 has non-negative charge. But how about faces of size 9? Suppose that f is a 9-face and has 8 bad vertices $v_1, \ldots, v_8$. Therefore, f is sending out all its $2|f| - 6 = 12$ units of charge to these bad vertices and has nothing left to send to its ninth vertex. In particular, if v_9, the ninth vertex of f, is a simple vertex, i.e. f is a simple face as in Figure 3.3(a), then f must send 1 unit of charge to v_9 and will have -1 charge. This inspires us to try to show that a simple face is in fact a reducible configuration.

In the next five lemmas, by a minimum-counter example we mean a graph $G \in \mathcal{G}_8$ which is is a counter-example to Theorem 3.1.1 with the minimum number of vertices, and that among those counter-examples which have the same number of vertices as G, G has the minimum number of edges.

Lemma 3.3.1 *A minimum counter-example cannot have a simple face.*

Proof: Let G be a minimum counter-example and suppose that f is a simple face in G. Let's denote the vertices of f by $v_1, v_2, \ldots, v_9$, in clockwise order, where $v_1, \ldots, v_8$ are bad and v_9 is simple. We denote the vertex adjacent to both v_{2i-1} and v_{2i} by w_i, $1 \leq i \leq 4$. The neighbour of v_9 not in the boundary of f is called w_5. (see Figure 3.6(a)). We modify G in the following way: remove $v_1, v_2, \ldots, v_9$ and their incident edges from G and add 6 new vertices $u_1, u_2, \ldots, u_6$. Make u_1, u_2, u_3 and u_4, u_5, u_6 two triangles and add the following edges: u_1w_1, u_2w_2, u_4w_3, u_5w_4, u_3u_6. (see Figure 3.6(b)).

Call this new graph G' and the new vertices and edges the *gadget*. Clearly G' is planar and it is straightforward to verify that the pairwise distances of $w_1, \ldots, w_5$ in G' using only the vertices and the edges of the gadget is not less than their corresponding distances in G using only the vertices and the edges that are removed. Thus $G' \in \mathcal{G}_8$. The number of vertices of G' is smaller than in G. So by minimality of G, there is a 3-colouring of

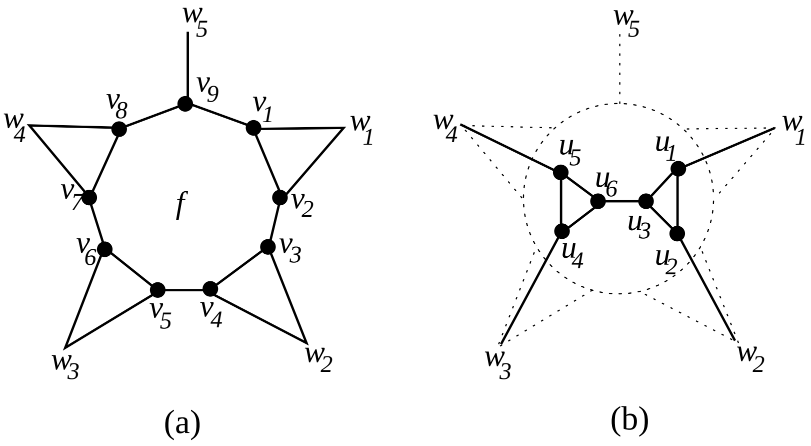


Figure 3.6: A simple face and the gadget added

G', called C. A very useful property of the gadget is that $w_1, \ldots, w_4$ cannot have all the same colour in C. We can easily prove this by contradiction. Assume that they all have got the same colour, say 1. Therefore, the colours of u_1, u_2, u_4, and u_5 are all different from 1. Since u_1, u_2, u_3 and u_4, u_5, u_6 are triangles and we are using only three colours in C, both u_3 and u_6 (which are adjacent) should have been coloured 1, which is impossible.

Consider colouring C induced on $G - \{v_1, \ldots, v_9\}$. The only coloured neighbour of v_9 is w_5. So we can extend C to v_9 by assigning a colour to it different from $C(w_5)$. Now the only two coloured neighbours of v_8 are w_4 and v_9, so there is a colour available for v_8. Using the same argument we can extend C by colouring $v_7, v_6, \ldots, v_2$, greedily. By the time we get to v_1 this greedy algorithm will assign a colour to v_1 different from $C(v_2)$ and $C(w_1)$. But since G is not 3-colourable, $C(v_1)$ must be equal to $C(v_9)$. Without loss of generality assume that $C(v_1) = C(v_9) = 1$. We could exchange $C(v_1)$ and $C(v_2)$ to resolve the conflict between $C(v_1)$ and $C(v_9)$, unless $C(v_3) = 1$. So assume that $C(v_3) = 1$. Similarly, we could exchange $C(v_3)$ and $C(v_4)$ to make $C(v_3) \neq 1$, unless $C(v_5) = 1$. So we must have $C(v_5) = 1$. By the same argument we can show that $C(v_7) = 1$.

Note: This technique is used by Sanders and Zhao [49]. We have already seen it in the proof of Claim 3.2.2 and will use it frequently in the proofs of other lemmas. We call this argument the "*chaining argument*".

On the other hand, without loss of generality, we can assume that $C(w_5) = 2$. Now if $C(v_8) = 2$ then we could simply assign $C(v_9) = 3$ and resolve the conflict between $C(v_9)$ and $C(v_1)$. We apply the chaining argument again. Therefore $C(v_8) = 3$ and $C(w_4) = 2$. If $C(v_6) \neq 3$ then we could simply exchange $C(v_7)$ with $C(v_8)$ and set $C(v_9) = 3$. Therefore $C(v_6) = 3$ and $C(w_3) = 2$. Using the same argument $C(v_4) = C(v_2) = 3$ and $C(w_2) = C(w_1) = 2$. But this means that all $w_1, \ldots, w_4$ have the same colour in C, a contradiction. [2] ■

Remark 3.3.2 *By this lemma, any 3-colouring of* $G - \{v_1, \ldots, v_9\}$ *in which not all* $w_1, \ldots, w_4$ *have the same colour can be extended to a 3-colouring of* G *in constant time. One way of doing this is using exhaustive search, considering all possible 3-colourings of* $v_1, \ldots, v_9$.

Continuing the discussions we had before Lemma 3.3.1, one other possibility for a 9-face f to have negative charge is that it has 8 bad vertices $v_1, \ldots, v_8$ and the ninth vertex of it, v_9, is a 4-vertex. In this case too, f sends 1 unit of charge to v_9, and therefore, has -1 charge. One might argue that if v_9 is a type 0 vertex it is incident with four non-triangular faces, and therefore, we might be able to change the discharging rules so that v_9 receives charge from the other faces and f does not have to send any charge to v_9. This saves 1 unit of charge for f and it will not have negative charge. This is a valid argument and in fact we do exactly that (see rule R5 in Section 3.3.3). But if v_9 is a type 2 vertex, i.e. if f is a type 2 face (as in Figure 3.7(a)), then there are only two non-triangular faces incident with v_9, one of them is f and let's call the other one f'. These are the only two non-triangle faces incident with v_9 and they should send charge to v_9. If f' too is a type 2 face, then each of f and f' must send 1 unit of charge to v_9, but

[2] The reader might have observed that this argument can be simplified by using the well-known fact that an odd cycle can be 2-list coloured as long as the lists are not all the same. But we prefer to use the above argument as we will generalize it to prove reducibility of some more complicated configurations.

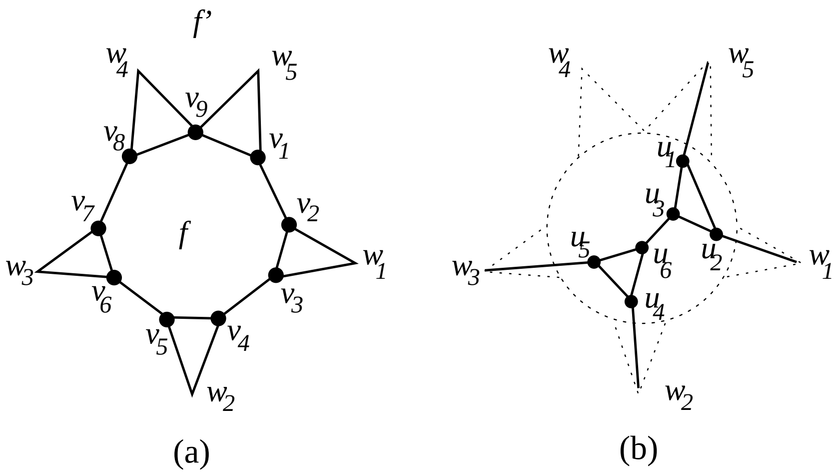


Figure 3.7: A type 2 face and the gadget added

they cannot afford to do so, as each of them is sending all of its charge to its bad vertices. So this is a situation in which we should be looking for a reducible configuration. In the following lemma we show that in fact a type 2 face (like f) is reducible.

Lemma 3.3.3 *A minimum counter-example cannot have a type 2 face.*

Proof: Let G be a minimum counter-example and suppose that f is a type 2 face of G whose bad vertices are $v_1, v_2, \ldots, v_8$ and whose type 2 vertex is v_9 (see Figure 3.7(a)). We modify G in a way similar to that of Lemma 3.3.1: remove $v_1, \ldots, v_9$ and add a gadget as in Figure 3.7(b).

It is straightforward to verify that the new graph G' is in $\mathcal{G}_8$, and by definition of G, there exists a 3-colouring of G', say C. Note that by the same argument as we had in Lemma 3.3.1, we cannot have all w_5, w_1, w_2, w_3 coloured with the same colour in C. Consider C induced on G. Since the only coloured neighbours of v_9 are w_4 and w_5, we can extend C to v_9. Assign a colour different from $C(v_9)$ and $C(w_5)$ to v_1. Also, starting from v_8 and moving around f toward v_2 in counter-clockwise order, we can extend C by colouring $v_8, \ldots, v_3$ greedily. We assign a colour different from $C(v_3)$ and $C(w_1)$ to v_2. Since G is 4-chromatic, v_2 will get the same colour as v_1, say 1. By the chaining argument $C(v_4) = C(v_6) = C(v_8) = 1$. Without loss of generality we assume $C(v_9) = 2$

which yields $C(w_4) = C(w_5) = 3$. If $C(v_7) \neq 2$ then we could set $C(v_8) = 2$, $C(v_9) = 1$, and $C(v_1) = 2$ and get a 3-colouring of G. So $C(v_7) = 2$ and $C(w_3) = 3$. By the chaining argument $C(v_5) = C(v_3) = 2$. This means that $C(w_1) = C(w_2) = C(w_3) = C(w_5) = 3$, which is a contradiction. ∎

Remark 3.3.4 *As in Remark 3.3.2, the proof of this lemma implies that any 3-colouring of $G - \{v_1, \ldots, v_9\}$, in which not all w_1, w_2, w_3, w_5 have the same colour, can be extended to a 3-colouring of G in constant time using exhaustive search.*

The previous two configurations are our only reducible configurations that involve only one 9-face. By extending the arguments preceding these two lemmas, we see that there are more complicated structures that we must prove are reducible. The general idea of the proof of all of the other configurations is basically the same as above. In most of them, we need to forbid some of the vertices from all having the same colour. To do this, we remove some vertices and edges of the minimum counter-example and add a gadget whose structure is similar to the one in the previous lemmas. In all the cases, the new graph is smaller and is in $\mathcal{G}_8$, hence is 3-colourable. Then we show that this 3-colouring induced on the original graph (which will be a partial 3-colouring), can be extended to a 3-colouring of the whole graph, contradicting an assumption that it is a minimum counter-example. This establishes the reducibility of the configuration.

The following lemma proves the reducibility of a configuration that involves two 9-faces, each of which is a type 0 face.

Lemma 3.3.5 *A minimum counter-example cannot have two type 0 faces sharing their type 0 vertex.*

Proof: Let G be a minimum counter-example and suppose that f_1 and f_2 are two type 0 faces in G sharing their type 0 vertex. There are two possible configurations of this kind (shown in Figures 3.8(a) and 3.9(a)). We consider each case separately:

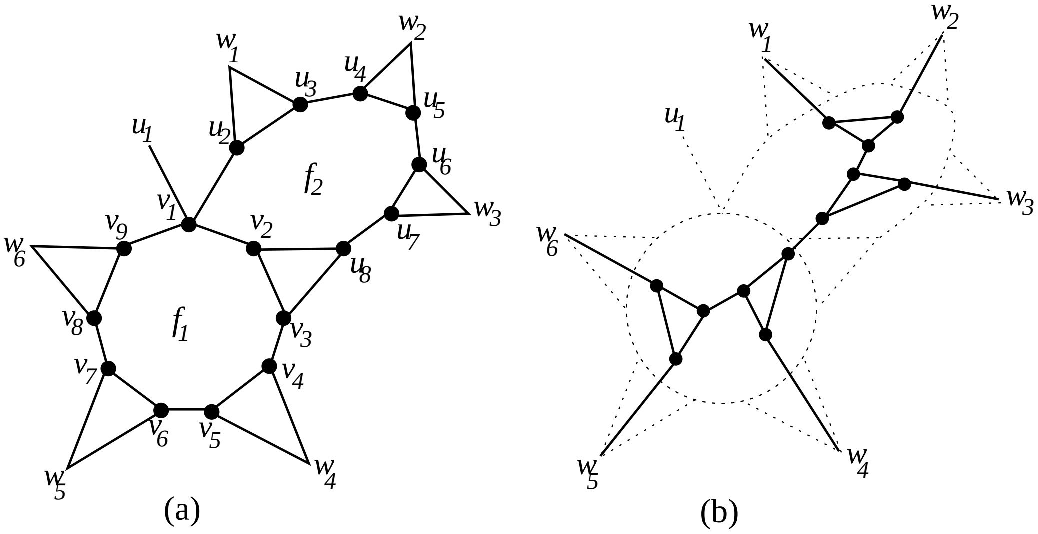


Figure 3.8: Two type 0 faces sharing a type 0 vertex

Configuration of Figure 3.8(a): First we remove $v_1, \ldots, v_9$ and $u_2, \ldots, u_8$ and all the incident edges. Then add four new triangles and connect them together and to the rest of the vertices of G as in Figure 3.8(b). Call this new graph G'. It is straightforward to verify that: (i) $G' \in \mathcal{G}_8$ (ii) because of minimality of G there is a 3-colouring of G', say C, and (iii) $w_1, \ldots, w_6$ cannot all have the same colour in C.

Now consider this 3-colouring induced on G. We can easily extend C to v_1, since only one neighbour of v_1, which is u_1, is coloured. Similarly, we can extend C by colouring $v_9, \ldots, v_3$ greedily. Also, starting from u_2 and moving around f_2 in clockwise order, we can colour $u_3, \ldots, u_8$, greedily. Now assign a colour different from $C(v_3)$ and $C(u_8)$ to v_2, which will be equal to $C(v_1)$. Without loss of generality, assume that $C(v_1) = C(v_2) = 1$. By the chaining argument starting from v_2 and going around f_1: $C(v_4) = C(v_6) = C(v_8) = 1$. Similarly, by the same argument for the vertices around f_2: $C(u_7) = C(u_5) = C(u_3) = 1$.

Without loss of generality assume $C(u_1) = 3$. Suppose that $C(u_2) = 3$. First exchange $C(v_3)$ with $C(u_8)$ (if needed) so that $C(v_3) \neq C(v_5)$. Now exchange $C(v_9)$ with $C(v_8)$, $C(v_7)$ with $C(v_6)$, and $C(v_5)$ with $C(v_4)$, and set $C(v_1) = 2$. This gives a 3-colouring of G which is a contradiction. Thus $C(u_2) = 2$ and by the chaining argument $C(u_4) =$

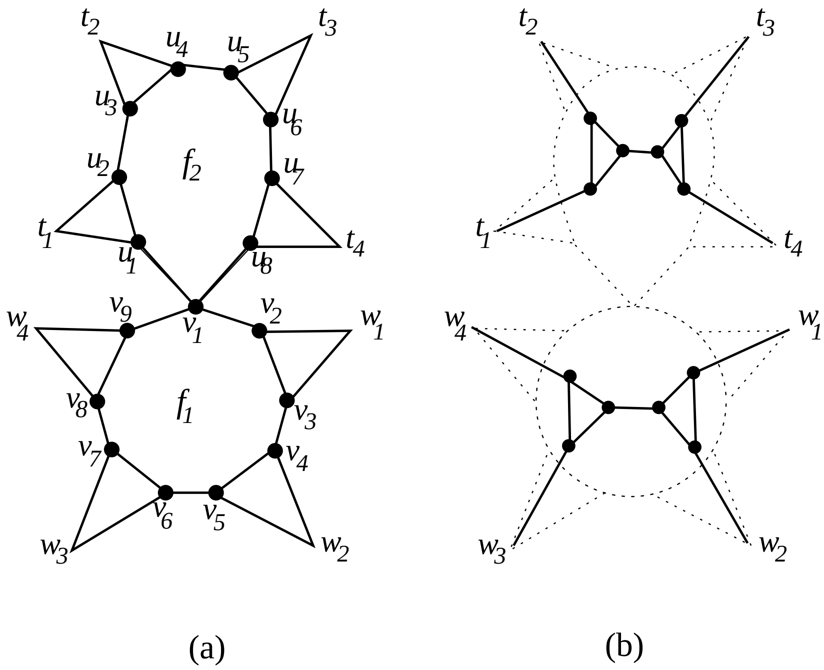


Figure 3.9: Two type 0 faces sharing a type 0 vertex

$C(u_6) = C(u_8) = 2$. Using exactly the same argument we can show that $C(v_9) = 2$ and by the chaining argument $C(v_7) = C(v_5) = 2$. But this means that $w_1, \ldots, w_6$ all have colour 3 in C, contradicting property (iii) mentioned for C.

Configuration of Figure 3.9(a): First remove $v_1, \ldots, v_9$ and $u_1, \ldots, u_8$ and all the incident edges. Then add four new triangles and connect them together and to the rest of the vertices of G as in Figure 3.9(b). Call this new graph G'. Again, it is straightforward to verify that: (i) $G' \in \mathcal{G}_8$, (ii) because of minimality of G there is a 3-colouring of G', say C, (iii) $w_1, \ldots, w_4$ cannot all have the same colour in C. Similarly, $t_1, \ldots, t_4$ cannot all have the same colour in C.

Now consider this 3-colouring induced on G. We extend C by colouring the uncoloured vertices of G greedily in the following order: $u_8, u_7, \ldots, u_1, v_1, v_9, v_8, \ldots, v_3$, since at each step there are at most two colours in the neighbourhood of the vertex we want to colour. We can also assign a colour different from $C(w_1)$ and $C(v_3)$ to v_2. By definition of G, $C(v_1) = C(v_2)$, which we can assume is equal to 1. By the chaining argument

$C(v_4) = C(v_6) = C(v_8) = 1$.

Without loss of generality, assume that $C(v_9) = 3$. We exchange $C(v_9)$ with $C(v_8)$. If $C(v_7) = 3$ exchange $C(v_7)$ with $C(v_6)$ and then if $C(v_5) = 3$ exchange $C(v_5)$ with $C(v_4)$. In this case $C(v_3)$ cannot be 3, otherwise all $w_1, \ldots, w_4$ have colour 2, contradicting (iii). Note that now $C(v_9) = C(v_1) = C(v_2) = 1$. We claim that $\{C(u_1), C(u_8)\} \neq \{2, 3\}$. By way of contradiction, assume that $C(u_1) = 2$ and $C(u_8) = 3$. If we could exchange $C(u_1)$ with $C(u_2)$, or $C(u_8)$ with $C(u_7)$, then there would be at most two colours in the neighbourhood of v_9, and therefore, we could assign a new colour to v_9 different from its neighbours, and get a 3-colouring of G. So, $C(u_3) = 2$ and by the chaining argument, $C(u_3) = C(u_5) = C(u_7) = 2$. Similarly, $C(u_8) = C(u_6) = C(u_4) = C(u_2) = 3$. But this implies that all $t_1, \ldots, t_4$ have colour 1 in C, contradicting property (iii) of C. ■

So in Lemma 3.3.5 we actually proved the reducibility of two subconfigurations. The same will be true for the next lemma. Our most complicated reducible configurations involve an interaction of three 9-faces. Next we prove the reducibility of one of them.

Lemma 3.3.6 *A minimum counter-example cannot have three type 5 faces sharing their 5-vertex.*

Proof: Let G be a minimum counter-example with three type 5 faces sharing their 5-vertex. There are two possible non-symmetric configurations of this kind, which are shown in Figures 3.10(a) and 3.11(a). We consider each case separately:

Configuration of Figure 3.10(a): First we remove $u_1, \ldots, u_7$, $v_2, \ldots, v_9$, $t_1, \ldots, t_7$, and all the incident edges. Then add 6 new triangles and connect them together and to the rest of the vertices of G as in Figure 3.10(b). Call this new graph G'. Again, it is straightforward to verify that: (i) $G' \in \mathcal{G}_8$, (ii) because of minimality of G there is a 3-colouring of G', say C, and (iii) $w_1, \ldots, w_8$ cannot all have the same colour in C.

Consider this 3-colouring induced on G. We extend C by colouring the uncoloured vertices of G greedily in the following order: $t_1, t_2, \ldots, t_7, v_9, v_8, \ldots, v_2, u_7, u_6, \ldots, u_2$, since

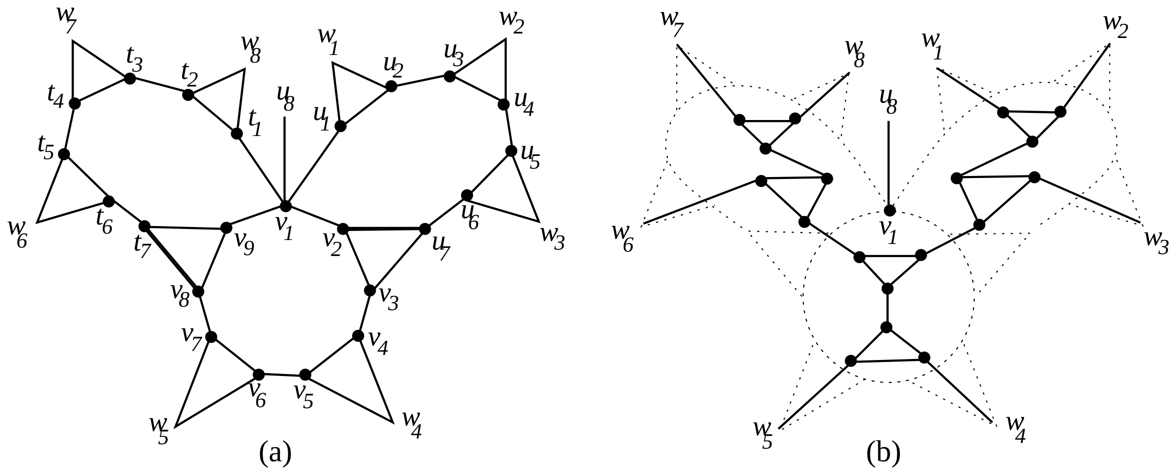


Figure 3.10: Three type 5 faces sharing a 5-vertex

at each step there are at most two colours in the neighbourhood of the vertex we want to colour. We also assign a colour different from $C(u_2)$ and $C(w_1)$ to u_1. Since G is not 3-colourable, $C(u_1) = C(v_1)$, which we can assume is equal to 1. By the chaining argument, $C(u_3) = 1$, and also all $u_5, u_7, v_4, v_6, v_8, t_6, t_4$, and t_2 must have been coloured 1. First we show that $C(t_1) \neq C(u_8)$. Assume that they are both equal, say 2. We can exchange $C(t_7)$ with $C(v_9)$ (if needed) so that $C(v_9) = 2$, too. Similarly, we can exchange $C(v_2)$ with $C(v_3)$ if needed to set $C(v_2) = 2$. Then we can set $C(v_1) = 3$ and get a 3-colouring of G.

So we can assume that $C(t_1) = 3$ and $C(u_8) = 2$. If we could exchange $C(t_1)$ with $C(t_2)$, by an argument similar to the previous case, we can set $C(v_9) = C(v_2) = 2$ and set $C(v_1) = 3$. This shows that we cannot exchange $C(t_1)$ with $C(t_2)$, because $C(t_3) = 3$. By the chaining argument $C(t_5) = 3$, too. Now, if $C(v_7) = 2$ then we could set $C(v_8) = 3$, $C(v_9) = 1$, $C(t_7) = 2$, and exchange $C(t_6)$ with $C(t_5)$, $C(t_4)$ with $C(t_3)$, and $C(t_2)$ with $C(t_1)$, and set $C(v_1) = 3$. This shows that $C(v_7) = 3$. By the chaining argument $C(v_5) = 3$, and by a similar argument we can show that $C(u_6) = C(u_4) = C(u_2) = 3$. All these show that $w_1, \ldots, w_8$ are all coloured with 2 which contradicts property (iii) mentioned above for C.

Configuration of Figure 3.11(a): First remove $v_1, \ldots, v_9, t_1, \ldots, t_8, u_1, \ldots, u_7$ and all

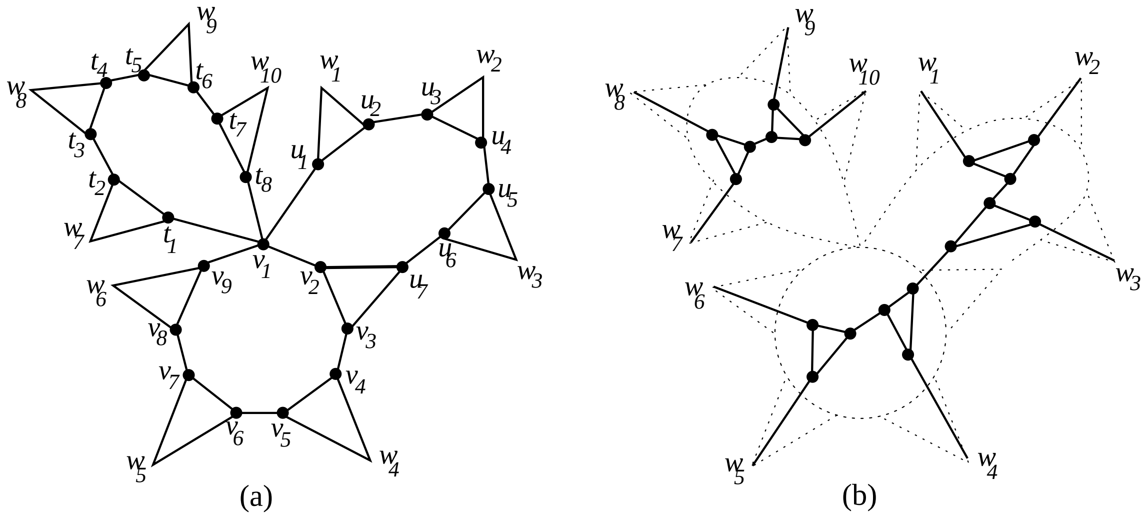


Figure 3.11: Three type 5 faces sharing a 5-vertex

the incident edges. Then add 6 new triangles and connect them together and to the rest of the vertices of G as in Figure 3.11(b). Call this new graph G'. It is straightforward to verify that: (i) $G' \in \mathcal{G}_8$, (ii) because of minimality of G there is a 3-colouring of G', say C, and (iii) $w_1, \dots, w_6$ cannot all have the same colour in C. Also, $w_7, \dots, w_{10}$ cannot all have the same colour in C.

Consider this 3-colouring induced on G. We extend C by colouring the uncoloured vertices of G greedily in the following order: $t_8, t_7, \dots, t_1, v_1, v_9, v_8, \dots, v_2, u_7, u_6, \dots, u_2$, since at each step there are at most two colours in the neighbourhood of the vertex we want to colour. We also assign a colour different from $C(u_2)$ and $C(w_1)$ to u_1. Since G is not 3-colourable, $C(u_1) = C(v_1)$, which we can assume is equal to 1. By the chaining argument, $C(u_3) = 1 = C(u_5) = C(u_7) = C(v_4) = C(v_6) = C(v_8)$.

First we show that $C(t_1) \neq C(t_8)$. By contradiction assume that they are equal to 2. So $C(v_9) = 3$, otherwise we could simply set $C(v_1) = 3$ and exchange $C(v_2)$ with $C(v_3)$ if needed. By the chaining argument $C(v_7) = C(v_5) = 3$. By the chaining argument, $C(u_6) = 3 = C(u_4) = C(u_2)$. But this requires that all $w_1, \dots, w_6$ be coloured 2, which contradicts property (iii) mentioned above.

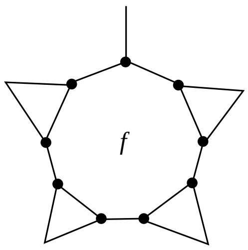
So we can assume that $C(t_1) = 2$ and $C(t_8) = 3$. If we could exchange $C(t_8)$ with

$C(t_7)$ then we could use the same argument as in the previous paragraph to modify C so that there are only colours 1 and 2 in the neighbourhood of v_1 and set $C(v_1) = 3$ to get a 3-colouring of G. This contradiction shows that $C(t_6) = 3$, and by the chaining argument $C(t_4) = C(t_2) = 3$. We can do a very similar argument to show that $C(t_3) = 2$ and by the chaining argument $C(t_5) = C(t_7) = 2$. But then we have to have $C(w_7) = C(w_8) = C(w_9) = C(w_{10}) = 1$ which contradicts property (iii) we mentioned. ■

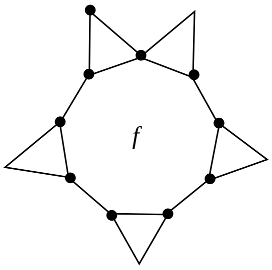
In addition to the reducible configurations we used in the proof of Theorem 3.2.1, we have seen four new configurations described in Lemmas 3.3.1 to 3.3.6, some of which have two subconfigurations. There are 8 other configurations. Below we list all these fifteen configurations, including the four we proved above and the three used in the proof of Theorem 3.2.1 (see Figure 3.12):

Reducible Configurations:

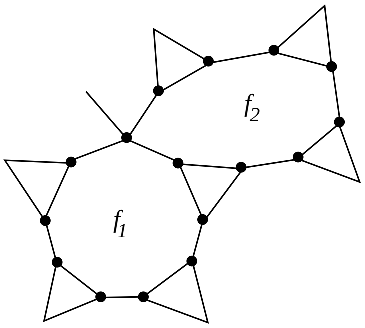
1. A $\leq$2-vertex.
2. A cut-vertex.
3. A $2k$-face with at least $2k-1$ bad vertices.
4. A simple face.
5. A type 2 face.
6. Two type 0 faces sharing their type 0 vertex.
7. Three type 5 faces sharing their type 5 vertex.
8. Two semi-simple faces sharing a type 1 vertex.
9. Two semi-type 2 faces sharing a type 1 vertex.
10. A semi-type 2 face sharing its type 1 vertex with a type 1 face.
11. A semi-type 2 face sharing its type 1 vertex with a semi-simple face.



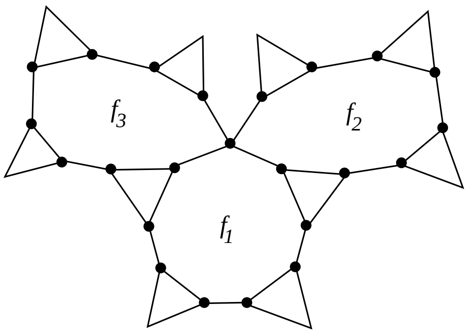
4: A simple face



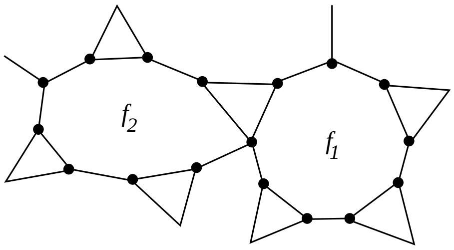
5: A type 2 face



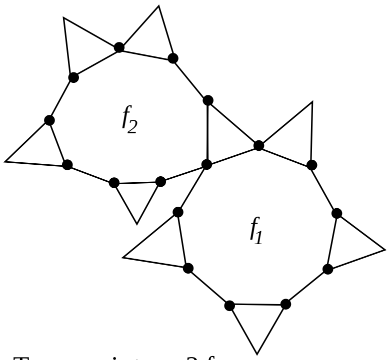
6: Two type 0 faces



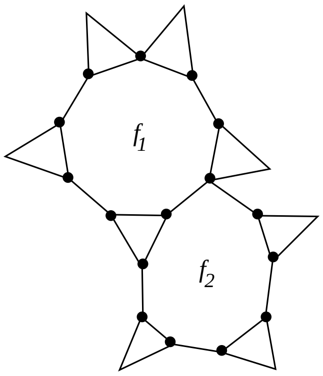
7: Three type 5 faces



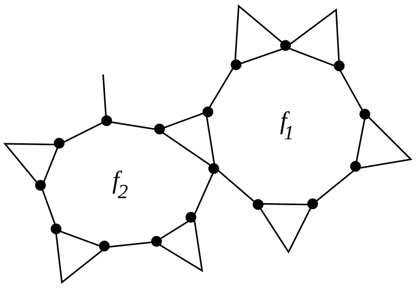
8: Two semi−simple faces



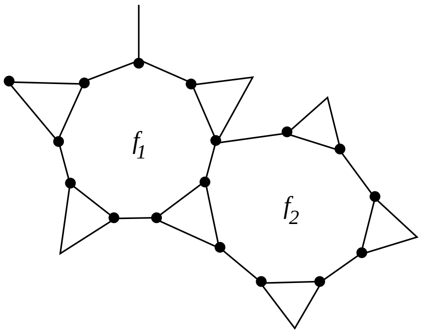
9: Two semi−type 2 faces



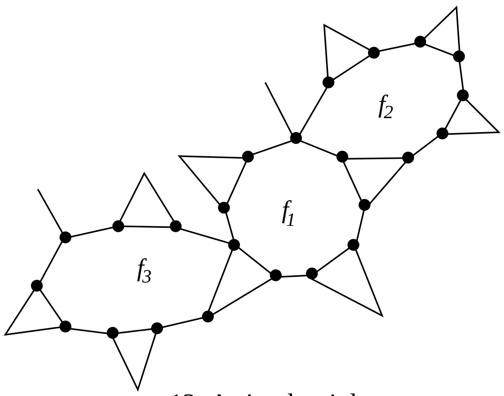
10: A semi−type 2 and a type 1 faces



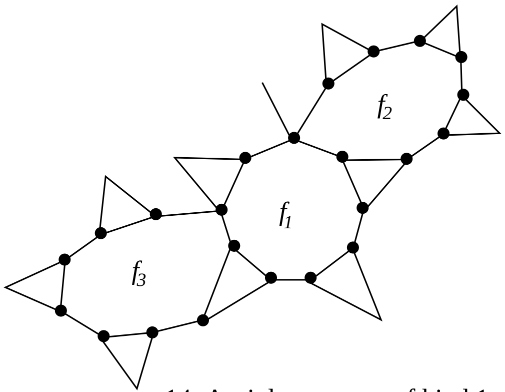
11: A semi−type 2 and a semi−simple face



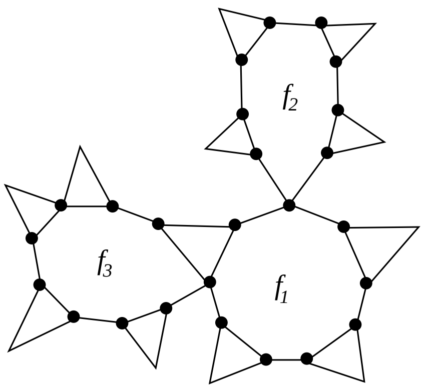
12: A semi−simple and a type 1 face



13: A simple triple structure



14: A triple structure of kind 1



15: A triple structure of kind 2

Figure 3.12: Reducible configurations 4-15

12. A semi-simple face sharing its type 1 vertex with a type 1 face.

13. A simple triple structure.

14. A triple structure of kind 1.

15. A triple structure of kind 2.

While this list has only 15 configurations, some of them (like configurations 6 and 7) have two subconfigurations, and some of them (like configurations 14 and 15) have many more subconfigurations, so many so that the total number of configurations (considering all subconfigurations) is 77.

Lemma 3.3.7 *A minimum counter-example to Theorem 3.1.1 cannot have any of the configurations given above.*

We have seen the hand-checkable proofs for configurations 1-7 (in the proofs of Example 2.2.3, Theorem 3.2.1, and Lemmas 3.3.1 to 3.3.6). We defer the proof of other configurations until Section 3.5 and Appendices A, B, and C, where we describe the hand-checkable proofs of configurations 8-12 and we discuss the computer-aided proof of all configurations.

Remark 3.3.8 *As in Remarks 3.2.3, 3.3.2, and 3.3.4, for each of the configurations given above, the proof of reducibility yields a constant time algorithm for extending a 3-colouring of the graph obtained by removing the vertices of the configuration (and possibly adding a gadget to it) to a 3-colouring of the original graph. One way of doing this is exhaustive search, i.e. considering all possible 3-colourings of the vertices of the configuration. Since each configuration has constant size this takes* $O(1)$ *time.*

We complete the proof of Theorem 3.1.1 by proving the unavoidability of these configurations (using the Discharging Method) in the next section.

3.3.3 Discharging Rules

Let G be an arbitrary graph in $\mathcal{G}_8$. As in the proofs of Theorem 3.2.1 and Example 2.2.4, we give an initial charge of $d(v) - 6$ units to each vertex v and $2|f| - 6$ units to each face f. By Euler's formula, the total charge is -12. In the discharging rules, we move some charges from faces to vertices. So the vertices do not lose any charge in the discharging phase.

Let's try the discharging rule we had in the proof of Theorem 3.2.1. That is, assume every non-triangle face f sends $\frac{3}{2}$ to each of its bad vertices and 1 unit to every other vertex. By this rule, as we proved in Example 2.2.4, every ≥ 11-face will have non-negative charge or else we have reducible configuration 1 or 2. Also, the only 10-faces with negative charge are those that have at least 9 bad vertices. But these faces are reducible (configuration 3). Therefore, we can keep this rule for ≥ 10-faces:

R1: Every ≥ 10-face sends $\frac{3}{2}$ to each of its bad vertices and 1 to each of its non-bad vertices.

If we use the same rule for 9-faces, there are several possible 9-faces that will have negative charge. For example, if a 9-face f is incident with 8 bad vertices and a type 0 vertex (i.e. f is a type 0 face), then f sends $\frac{3}{2} \times 8 = 2|f| - 6$ units of charge to the bad vertices and it cannot afford to send another 1 unit of charge to its non-bad vertex. Some but not all such situations are dealt with using new reducible configurations introduced in the previous subsection. For others, we need to modify the discharging rule.

If a 9-face is incident with at most 6 bad vertices, then it has to send at most $\frac{3}{2} \times 6 = 9$ units to them, and can afford to send 1 unit of charge to every other vertex in its boundary. So we can keep our standard rule for such a 9-face. Also, recall from the proof of Example 2.2.3 that, since there are no 4-cycles, every ≥ 5-vertex is incident with at least two non-triangle faces. If v is a 5-vertex (the largest degree with negative initial charge), it has initially -1 charge and it only needs to get at most $\frac{1}{2}$ unit from each of the at least two non-triangle faces that are incident with it. Therefore, if a 9-face is incident with 7 bad vertices and at least one ≥ 5-vertex, then it can send $\frac{3}{2} \times 7 = 10.5$ to the bad vertices, $\frac{1}{2}$ to the 5-vertex (if there is one), and 1 to the other vertex. We combine these into the following rule:

> **R2:** If f is a 9-face with at most 6 bad vertices, or with exactly 7 bad vertices and at least one ≥ 5-vertex, then f sends $\frac{3}{2}$ to each of its bad vertices, 1 to each of its 4-vertices, and $\frac{1}{2}$ to each of its 5-vertices.

Now we prove that by these two rules, every ≥ 5-vertex either has non-negative charge after the discharging phase or lies in a reducible configuration.

Lemma 3.3.9 *Every $\geq$5-vertex will either have non-negative charge, after the discharging phase, or lie in a reducible configuration*

Proof: If $d(v) \geq 6$ then its initial charge is non-negative and it doesn't lose any charges in the discharging phase. Assume that $d(v) = 5$ and the faces incident with v in clockwise order are f_1, f_2, f_3, f_4, f_5. Note that either all these faces are distinct or v is a cut-vertex (reducible configuration 2). Recall that a type 5 face is a 9-face incident with 8 bad vertices and a 5-vertex. If none of $f_1, \ldots, f_5$ is a triangle then at least three of them are type 5 faces or at least three of them are not type 5 faces. In the former case, G has reducible configuration 7. In the latter case by rules R1 or R2 each of the three sends at least $\frac{1}{2}$ to v and so v will have non-negative charge.

Assume that exactly one of $f_1, \ldots, f_5$, say f_1, is a triangle. Then f_2 and f_5 are not type 5 and so each one is either a ≥ 10-face or a 9-face with at most 7 bad vertices. Thus, each sends $\frac{1}{2}$ to v, by rules R1 or R2.

Finally, assume that exactly two of $f_1, \ldots, f_5$ are triangles (if more than two of them are triangles then G will have a 4-cycle). Note that these triangles cannot be adjacent because G cannot have a 4-cycle. Without loss of generality, assume that f_1 and f_3 are

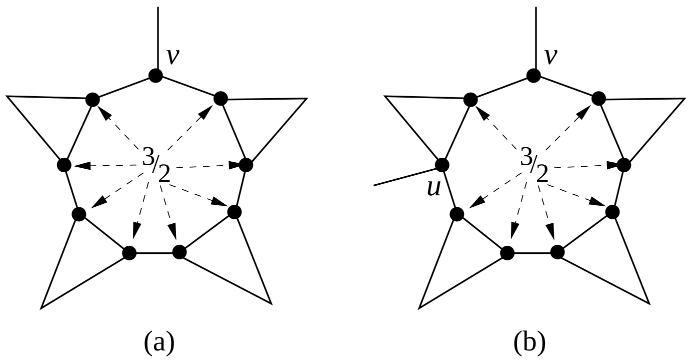


Figure 3.13: (a) A simple vertex v incident with a simple face f (b) a simple vertex v incident with a semi-simple face f

triangles. Thus, f_4 and f_5 cannot be of type 5, and as in the previous case, each of them sends at least $\frac{1}{2}$ to v by rules R1 or R2. ■

So the only vertices that remain to be dealt with are 3- and 4-vertices. Remember that 3-vertices (with initial charge of -3) are the most desperate vertices for charge. As we have argued before, every bad vertex should get $\frac{3}{2}$ from the non-triangular faces that are incident with it. Thus, in all our rules we insist that every 9-face sends $\frac{3}{2}$ to each of its bad vertices. Also, if v is a simple vertex, it needs to get 3 units of charge from the three faces it is incident with. Each of these faces sends 1 unit of charge to v by the rules given so far, if it is a $\geq$10-face or a 9-face with at most 6 bad vertices, or a 9-face incident with 7 bad vertices and a $\geq$ 5-vertex. What if some of these three faces are 9-faces to which rule R2 does not apply? For example, if f is a 9-face incident with simple vertex v and 8 bad vertices (see Figure 3.13(a)), then f must send $\frac{3}{2} \times 8 = 2|f| - 6$ to its bad vertices and it has nothing left to send to v. This is why we proved in Lemma 3.3.1 that a face like f (a simple face) is reducible. Thus, if G has such a configuration we are done.

To complete our analysis of the 3-vertices, the only other possibility we have to consider is when f is a 9-face with 7 bad vertices, one simple vertex (which is v) and the other vertex u is a $\leq$4-vertex (since otherwise rule R2 applies to f). This is possible only if u is a type 1 vertex, i.e. f is a semi-simple face (see Figure 3.13(b)). In this case,

f must send $\frac{3}{2} \times 7 = 10.5$ to its bad vertices and, as we discussed above, it has to send 1 unit to its simple vertex, v. Therefore, it has only $\frac{1}{2}$ unit of charge left to be sent to its type 1 vertex, u. We hope that since u is incident with two other non-triangle faces, it can receive enough charge from them to have non-negative charge. So, for the moment, let's assume that u will be fine. We will deal with it later. Thus, every 9-face that has a simple vertex (other than a simple face which is reducible configuration 4) can afford to send 1 unit of charge to it. This way, we are sure that every 3-vertex, whether it is bad or simple, gets 3 units of charge from the faces incident with it and will have non-negative charge. So we introduce the following rule:

> **R3:** Every 9-face sends $\frac{3}{2}$ to each of its bad vertices and 1 unit of charge to each of its simple vertices.

Note that if f is semi-simple (as in Figure 3.13(b)), by the above rule it sends out $\frac{3}{2} \times 7 + 1 = 11.5$ units, and still has $\frac{1}{2}$ units of charge. Later, we will give a rule to make use of this charge by moving it from f to its type 1 vertex, u.

Since in R3 we say *every* 9-face sends $\frac{3}{2}$ units to each of its bad vertices, it is redundant to say in R2 that every 9-face with at most 6 bad vertices, or with 7 bad vertices and at least one $\geq$5-vertex sends $\frac{3}{2}$ to its bad vertices. So we can modify R2 as follows:

> **New R2:** If f is a 9-face incident with at most 6 bad vertices, or with exactly 7 bad vertices and at least one $\geq$5-vertex, then f sends 1 to each of its 4-vertices, and $\frac{1}{2}$ to each of its 5-vertices.

Therefore, these three rules ensure that every 3-vertex has non-negative charge, or it is a cut-vertex (reducible configuration 1) or in a simple face (reducible configuration 4). Thus, we have proved:

Lemma 3.3.10 *Each 3-vertex will either have non-negative charge after the discharging phase or lie in a reducible configuration.*

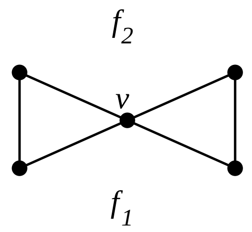


Figure 3.14: A type 2 vertex v

By these three rules, we also know that, so far, all ≥ 9-faces that do not lie in reducible configurations have non-negative charge and in many cases they have positive charge. So the only elements with negative charge that we have to deal with are 4-vertices. By the first rule we know that 4-vertices are getting 1 unit of charge from every ≥ 10-face that they are incident with. So the remaining cases we have to consider are incidences of 4-vertices with 9-faces. The rest of the rules we introduce here are for dealing with these cases, by moving the remaining positive charge on 9-faces to degree 4 vertices.

If every non-triangle face (including every 9-face) could send 1 unit of charge to its ≥ 4-vertices, then by the arguments of the proof of Theorem 3.2.1, all ≥ 4-vertices would have non-negative charge, too. But the problem is that 9-faces cannot necessarily afford to do this. For example, if a 9-face has 7 bad vertices and two 4-vertices, it sends $\frac{3}{2} \times 7 = 10.5$ units to its bad vertices by the third rule above, and it has only $\frac{3}{2}$ units of charge left for its two 4-vertices. Therefore, some 9-faces can only afford to send 1 unit of charge to one of their 4-vertices and at most $\frac{1}{2}$ unit of charge to the other one.

Recall that there are only three kinds of 4-vertices: type 0, type 1, and type 2. Assume that v is a type 2 vertex, incident with two triangles and two non-triangle faces f_1 and f_2 (See Figure 3.14). Note that $f_1 \neq f_2$, or else v is a cut-vertex (reducible configuration 2) and we are done. Since f_1 and f_2 are the only non-triangle faces incident with v, they should provide the 2 units of charge that v needs. If each of f_1 and f_2 is a ≥ 10-face, or a 9-face that has at most 6 bad vertices, or a 9-face with 7 bad vertices and a ≥ 5-vertex, then each sends 1 unit of charge to v by R1 or R2 and v will have non-negative charge.

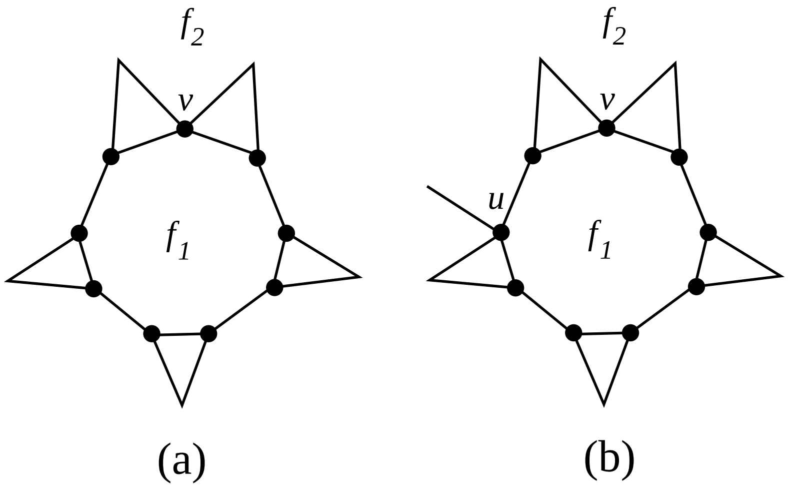


Figure 3.15: (a) f_1 is incident with 8 bad vertices and a type 2 vertex, (b) f_1 is a semi-type 2 face incident with v

Problems may arise when none of R1 or R2 applies to f_1, or none of R1 or R2 applies to f_2. Without loss of generality, let's assume that none of R1 or R2 applies to f_1. This implies that f_1 is a 9-face with at least 7 bad vertices.

If f_1 is a 9-face with 8 bad vertices and a type 2 vertex, v (see Figure 3.15(a)), then f_1 sends $\frac{3}{2} \times 8 = 2|f| - 6$ units to its bad vertices by R3 and cannot afford to send anything to v. This is why we proved in Lemma 3.3.3 that this configuration, i.e. a type 2 face, is reducible (configuration 5).

If f_1 contains 7 bad vertices and R2 does not apply to it, then f_1 contains two 4-vertices, one of which is v (a type 2 vertex), and the other is a type 1 vertex, say u. In other words, f_1 is a semi-type 2 face (See Figure 3.15(b)). In this case, f_1 sends $\frac{3}{2} \times 7 = 10.5$ to its bad vertices by R3 and must send 1 unit to v. So it will be left with only $\frac{1}{2}$ unit to be sent to u (its type 1 vertex). As before, we hope that since u is incident with three non-triangle faces, it will receive enough charge from the other faces it is incident with, so much so that it too will have non-negative charge. So we introduce the following rule:

R4: If f is a semi-type 2 face then it sends 1 unit of charge to its type 2

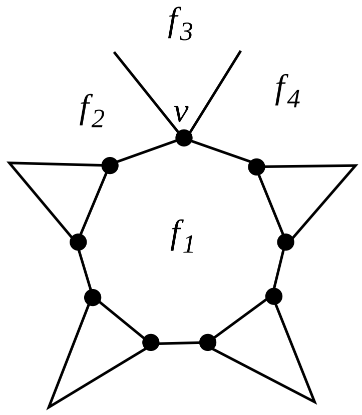


Figure 3.16: (a) A type 0 vertex v and the four faces around it

vertex and $\frac{1}{2}$ unit of charge to its type 1 vertex.

By this rule, we ensure that all type 2 vertices have non-negative charge, and that no 9-face has negative charge, unless it contains or lies in one of reducible configurations 1, 2, 4, 5, or 7. So, the only 4-vertices which still concern us are type 0 and type 1 vertices.

Consider a type 0 vertex v, i.e. one that is incident with four faces f_1, f_2, f_3, and f_4 (see Figure 3.16), where none of these faces is a triangle. These faces are all distinct, otherwise v is a cut-vertex (reducible configuration 2) and we are done. If we can prove that v receives at least $\frac{1}{2}$ unit of charge from each of $f_1, \ldots, f_4$ then it will have non-negative charge. This definitely happens if each of $f_1, \ldots, f_4$ is a ≥ 10-face, or a 9-face with at most 6 bad vertices, or a 9-face with 7 bad vertices and at least one ≥ 5-vertex, as each of f_1, f_2, f_3, f_4 sends 1 unit of charge to v by R1 or R2. Even if some (or all) of f_1, f_2, f_3 and f_4 are 9-faces with 7 bad vertices and two ≤ 4-vertices (one of which is v), then they send $\frac{3}{2} \times 7 = 10.5$ to their bad vertices and can afford to send $\frac{1}{2}$ to v and 1 to their other 4-vertex. In this case too v gets at least $4 \times \frac{1}{2}$ units and will have non-negative charge.

The only possible problem is when at least one of f_1, f_2, f_3, f_4, say f_1, cannot afford to send even $\frac{1}{2}$ to v. This happens only if f_1 is a 9-face with 8 bad vertices, i.e. it is a type 0 face. In this case f_1 sends $\frac{3}{2} \times 8 = 2|f_1| - 6$ units to its bad vertices and has nothing to contribute to v. But if f_1 is type 0 then none of f_2, f_3, f_4 can be a type 0 face, or else G has reducible configuration 6 and we are done. So each of f_2, f_3, f_4 can afford to send at least $\frac{1}{2}$ unit of charge to v. If at least one of them sends at least 1 unit of charge to v (by R1 or R2) then v has non-negative charge. This does not happen only if each of f_2, f_3, and f_4 is a 9-face with 7 bad vertices and a 4-vertex (other than v), which is a type 1 vertex, i.e. it is a semi-type 0 face. In this case, each of them can only afford to send $\frac{3}{4}$ units of charge to each of its 4-vertices. Again, we hope that for each face f_2, f_3, f_4 the other 4-vertex, which is a type 1 vertex, receives enough charge from the other faces to have non-negative charge. Therefore, we add the following to our bag of discharging rules:

> **R5:** If f is a semi-type 0 face with a type 0 vertex v which is not incident with a type 0 face, then f sends $\frac{1}{2}$ to v and 1 unit to its type 1 vertex. If v is incident with a type 0 face (like f_1 above), then f sends $\frac{3}{4}$ to v and $\frac{3}{4}$ to its type 1 vertex.

This ensures that every type 0 vertex not lying in a reducible configuration will have non-negative charge. Also, no face will end up with negative charge unless it is in a reducible configuration. Thus, with the discharging rules we have given so far:

Lemma 3.3.11 *Every type 0 or type 2 vertex will either have non-negative charge after the discharging phase, or lie in a reducible configuration.*

So the only 4-vertices with possible negative charge are type 1 vertices. By following similar arguments we develop two other discharging rules (rules R6 and R7 below), which ensure that if a type 1 vertex has negative charge then it is in a reducible configuration. We design these two new rules to make sure that 9-faces that are not in a reducible configuration have non-negative charge too. We can summarize the discharging rules as:

R1: Every ≥ 10-face sends $\frac{3}{2}$ to each of its bad vertices and 1 to each of its non-bad vertices.

R2: If f is a 9-face incident with at most 6 bad vertices, or with exactly 7 bad vertices and at least one ≥ 5-vertex, then f sends 1 to each of its 4-vertices, and $\frac{1}{2}$ to each of its 5-vertices.

R3: Every 9-face sends $\frac{3}{2}$ to each of its bad vertices and 1 unit of charge to each of its simple vertices.

R4: If f is a semi-type 2 face then it sends 1 unit of charge to its type 2 vertex and $\frac{1}{2}$ unit of charge to its type 1 vertex.

R5: If f is a semi-type 0 face with a type 0 vertex v which is not incident with a type 0 face, then f sends $\frac{1}{2}$ to v and 1 unit to its type 1 vertex. If v is incident with a type 0 face (like f_1 above), then f sends $\frac{3}{4}$ to v and $\frac{3}{4}$ to its type 1 vertex.

R6: If f is semi-simple then it sends $\frac{1}{2}$ units to its type 1 vertex.

R7: If f is semi-type 1 then it sends 1 unit to its type 1 vertex which is incident to a triangle that shares an edge with f, and sends $\frac{1}{2}$ to its other type 1 vertex.

An important observation, that will be helpful in the rest of the proof, is:

Observation 3.3.12 *Every non-triangle face sends at most 1 to each of its ≥ 4-vertices.*

We have already established that any 9-face to which only R2-R5 apply has non-negative charge, unless it is in or contains a reducible configuration. The only remaining 9-faces to consider are those to which R6 or R7 apply.

If R6 applies to a face f then f is semi-simple. So, it has 7 bad vertices and sends $7 \times \frac{3}{2} = 10.5$ to them by R3 and $1 + \frac{1}{2}$ to its 4-vertices by R6, for a total of 12, and no other rule applies to f. If R7 applies to face f then f is semi-type 1 and sends $7 \times \frac{3}{2} = 10.5$ to its 7 bad vertices by R3 and sends $1 + \frac{1}{2}$ to its 4-vertices by R7, for a total of 12, and no other rule applies to f. Thus, we have:

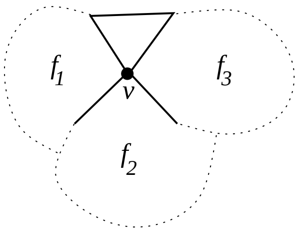


Figure 3.17: A type 1 vertex v

Lemma 3.3.13 *Each 9-face f will either have non-negative charge after the discharging phase, or lie in or contain a reducible configuration.*

The only remaining elements to consider are type 1 vertices. We prove that each type 1 vertex either lies in a reducible configurations listed in the previous subsection, or has non-negative charge after the discharging phase.

Lemma 3.3.14 *Every type 1 vertex v will either have non-negative charge after the discharging phase, or lie in a reducible configuration.*

Proof: Since the initial charge of v is -2 it is enough to show that during the discharging phase v gets at least 2 units of charge. Label the non-triangle faces incident with v: f_1, f_2, and f_3. (see Figure 3.17).

Note that f_1 and f_3 cannot be 9-faces with 8 bad vertices, because v is a type 1 vertex for each of them that is incident with a triangle that shares an edge with each of them. Therefore, f_1 and f_3 cannot be simple, type 0, type 1, or type 2. So each of f_1 and f_3 can only be a $\geq$10-face or a 9-face with at most 7 bad vertices.

If at least two of f_1, f_2, f_3 send 1 unit to v, then v has non-negative charge. So let's assume that at least two of them each send less than 1 unit of charge to v. This implies that at least one of f_1 or f_3 is sending less than 1 unit of charge. Without loss of generality, assume it is f_1 (by symmetry, the same arguments work for f_3). Thus rules R1 and R2 do not apply to f_1. Thus, since we said f_1 cannot have 8 bad vertices, f_1 has exactly 7 bad vertices and has no $\geq$5-vertex. Also, f_1 cannot be semi-type 1, by the assumption that it is sending less than 1 unit of charge to v and by rule R7.

Therefore, f_1 is either (1) semi-simple, (2) semi-type 0, or (3) semi-type 2. f_3 can be either of the following: (1) a $\geq$10-face, (2) a 9-face with at most 6 bad vertices, (3) semi-simple, (4) semi-type 0, (5) semi-type 1, or (6) semi-type 2. We consider different cases based on the types of f_1 and f_3:

- f_1 *is semi-simple:* So f_1 sends $\frac{1}{2}$ to v by R6. Since f_1 is semi-simple, if f_2 is of type 1, then G has reducible configuration 12. Otherwise f_2 sends at least $\frac{1}{2}$ to v, by rules R1, R2, or R7. It is enough to show that either f_3 sends at least 1 unit to v or G has a reducible configuration. We consider different possible cases for f_3:

  - $\geq$10-face: Sends 1 unit to v by R1.
  - 9-face with at most 6 bad vertices: Sends 1 unit to v by R2.
  - semi-simple: Since f_1 is semi-simple then G has reducible configuration 8.
  - semi-type 0: It sends 1 unit of charge by rule R5, unless its type 0 vertex is incident with a type 0 face, say f_4, in which case it only sends $\frac{3}{4}$ to v by R5. But, in that case f_3, f_4, and f_1 form a simple triple structure (see Figure 3.18(a)), which is reducible configuration 13.
  - semi-type 1: It sends 1 unit to v by R7.
  - semi-type 2: Since f_1 is semi-simple then G has reducible configuration 11.

- f_1 *is semi-type 0:* So f_1 sends $\frac{3}{4}$ to v by R5 (since we assumed it sends less than 1 unit to v). This implies that it is adjacent to a type 0 face, say f_4. If f_2 is a type 1 face then f_1, f_2, and f_4 form a triple structure of kind 1 (see Figure 3.18(b)), which is reducible configuration 14. Otherwise, f_2 sends at least $\frac{1}{2}$ to v by R1, R2, or R7. So v receives a total of at least $\frac{3}{4} + \frac{1}{2}$ from f_1 and f_2. It is enough to show that it receives at least $\frac{3}{4}$ from f_3 or G has a reducible configuration. We consider different possible cases for f_3.

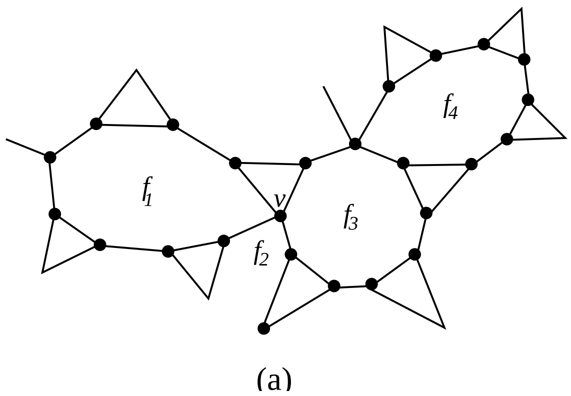


Figure 3.18: (a) f_1 is semi-simple, f_3 semi-type 0, and f_4 type 0 (b) f_1 is semi-type 0, f_2 type 1, and f_4 type 0

- $\geq$10-face: Sends at least 1 unit to v by R1.

- 9-face with at most 6 bad vertices: Sends at least 1 unit to v by R2.

- semi-simple: Then f_1, f_4 and f_3 form a simple triple structure (reducible configuration 13).

- semi-type 0: Then f_3 sends at least $\frac{3}{4}$ to v by R5.

- semi-type 1: It sends 1 unit of charge to v by rule R7.

- semi-type 2: Then f_1, f_4, and f_3 form a triple structure of kind 2 (reducible configuration 15).

- f_1 *is semi-type 2:* Thus f_1 sends $\frac{1}{2}$ to v by R4. Since f_1 is semi-type 2, if f_2 is of type 1, then G has reducible configuration 10. Therefore, f_2 sends at least $\frac{1}{2}$ to v by R1, R2, or R7. So v gets a total of at least 1 unit from f_1 and f_2. It is enough to show that f_3 sends at least 1 unit to v or G has a reducible configuration. We consider different cases based on the type of f_3:

  - $\geq$10-face: Sends 1 unit to v by R1.

  - 9-face with at most 6 bad vertices: Sends 1 unit to v by R2.

- semi-simple: Because f_1 is semi-type 2, if f_3 is semi-simple then they form reducible configuration 11.

- semi-type 0: If it is of a kind that sends $\frac{3}{4}$ to v by rule R5, then f_3 with its adjacent type 0 face (that is sharing the type 0 vertex of f_3), together with f_1 form a triple structure of kind 2 (reducible configuration 15). Otherwise it sends 1 unit to v.

- semi-type 1: Then it sends 1 unit to v by rule R7.

- semi-type 2: If f_3 is semi-type 2 then G has reducible configuration 9.

■

Proof of Theorem 3.1.1: By Lemmas 3.3.9, 3.3.10, 3.3.11, 3.3.13, and 3.3.14 either G has a reducible configuration listed in the previous subsection, or all the elements of G have non-negative charge, after applying the discharging rules. The latter is impossible, since the total initial charge is -12. So every graph $G \in \mathcal{G}_8$ has one of the reducible configurations, which proves the non-existence of a minimal counter-example to the theorem.

■

3.4 A 3-Colouring Algorithm for Planar Graphs Without 4- to 8-Cycles

As for the proofs of Example 2.2.4 and Theorem 3.2.1, the proof of Theorem 3.1.1 yields a quadratic time algorithm that given an embedded graphs in $\mathcal{G}_8$ produces a 3-colouring of G. At each iteration of the algorithm, we find a reducible configuration, break the graph into smaller subgraphs or reduce the number of vertices or edges of the graph by at least one, find a colouring of the smaller graphs, and extend these colourings to the original graph. We keep doing this as long as the graph is non-empty. We assume that the input graph to our colouring procedure is connected, as for a disconnected graph it is enough to find a 3-colouring for each of its connected components.

More specifically, at each iteration we apply the initial charges and the discharging rules, as described in Section 3.3.3. Since the total charge is negative, there must be some element (face or vertex) with negative charge. If it is a face it must be a 10-face with at least 9 bad vertices, or a simple, or a type 2 face. If the element is a vertex, call it v, then by Lemmas 3.3.9, 3.3.10, 3.3.11, and 3.3.14, v must be a ≤ 2-vertex, or a cut-vertex, or a vertex of one of configurations 6-15. Therefore, in any of these two cases (a face with negative charge or a vertex with negative charge), we find one of the reducible configurations from our list. If the configuration is one of the first three configurations, we do as in the algorithm of Theorem 3.2.1. Otherwise, we construct a smaller graph $G' \in \mathcal{G}_8$, which is obtained by removing some vertices and edges, and possibly adding a gadget, according to the proof of that reducible configuration. Then we find a 3-colouring of G', recursively. By Remark 3.3.8 we can extend this 3-colouring to a 3-colouring of G, in constant time.

Applying the initial charges takes at most $O(|V| + |F|)$ time. For each face f, it takes constant time to find the rules that apply to face f and it takes $O(|f|)$ to apply them to f. So applying the discharging rules takes at most $O(\sum_{f \in F} |f|)$ time, which is in $O(|E|)$, and once we have done that, we can find an element with negative charge in $O(|V| + |F|)$ time. Finding a reducible configuration around an element with negative charge and constructing the graph G' from G (i.e. removing the vertices and edges and adding the gadget) takes at most constant time. Thus if we define the size of the graph, n, to be $|V| + |E|$, we can say all these steps take at most αn time, for some constant $\alpha > 0$.

Let's denote the worst case running time of the procedure for an input of size n by $T(n)$. As in the analysis of the algorithms of Example 2.2.4 and Theorem 3.2.1, we can use induction to prove that for all values of $n \geq 1$ and for some constant $C > 0$: $T(n) \leq Cn^2$. The inequality is trivial for small values of n. So let's assume that $T(i) \leq Ci^2$ for all

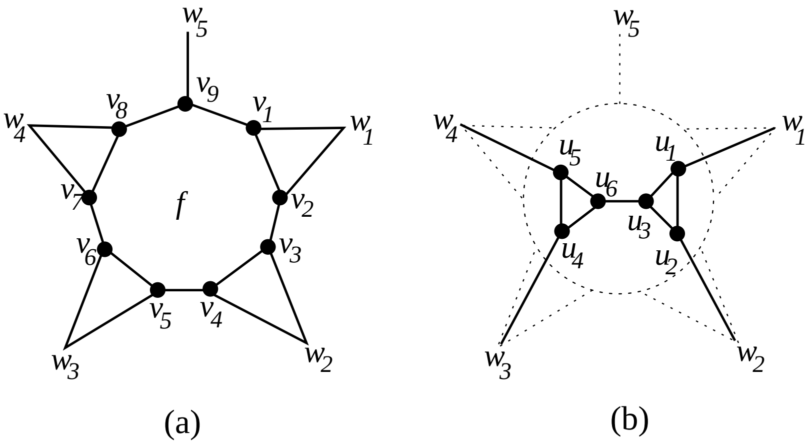


Figure 3.19: (a) A simple face and (b) the gadget

values of $1 \leq i < n$ and consider the procedure call when the input has size n.

If a 2-vertex or a cut-vertex or a face with negative charge is found, by an argument identical to that of the analysis of algorithm of Theorem 3.2.1 we can show that $T(n) \leq Cn^2$. If a vertex with negative charge is found and this vertex belongs to one of the configurations 6-15 then the algorithm makes a recursive call on the modified graph G', obtained according to the proof of that reducible configuration. Since G' has fewer vertices and/or edges with respect to G, the size of G', n', is smaller than n. Therefore $T(n) \leq \alpha n + T(n') \leq \alpha n + Cn'^2 \leq \alpha n + C(n-1)^2 \leq Cn^2$, for large enough C.

3.5 Automated Proof of the Reducible Configurations

Appendix A gives hand-checkable proofs for configurations 8-12, but it does not contain proofs for the last three configurations. Instead, we have an automated proof for all the configurations (See Appendix B). Here we describe how that proof works.

Consider the simple face of Figure 3.19(a). To prove that this is a reducible configuration, it is enough to check that every 3-colouring of the vertices $w_1, \ldots, w_5$, in which not all $w_1, \ldots, w_4$ have the same colour, can be extended to a 3-colouring of $v_1, \ldots, v_9$.

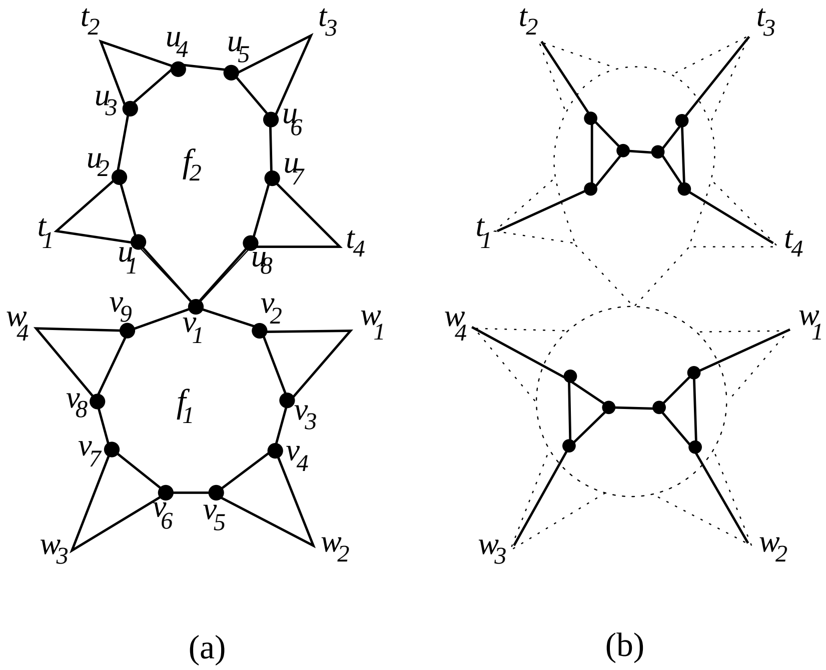


Figure 3.20: Configuration 6 which has two constrained groups

This easy task can be done by a simple program. The program generates all 3-colourings of $w_1, \ldots, w_5$ in which not all $w_1, \ldots, w_4$ have the same colour. For each such colouring C, since every vertex in $\{v_1, \ldots, v_9\}$ is adjacent to exactly one coloured vertex, there is a list of two colours available for every vertex in $\{v_1, \ldots, v_9\}$. Then the program uses exhaustive search to see if C can be extended to $v_1, \ldots, v_9$ using these lists. We have to do a similar job for each of the other configurations.

For any reducible configuration R, a vertex v which is not in R but has a neighbour in R is called a *boundary neighbour*. For example $w_1, \ldots, w_5$ in Figure 3.19(a) are boundary neighbours. For some configurations, such as a simple face, we have to forbid some of the boundary neighbours from all having the same colour. We do this by adding a gadget. We call this set of boundary neighbours a *constrained group*. For some reducible configurations (such as the configuration of Figure 3.20) we have two constrained groups. A 3-colouring of the boundary neighbours of a reducible configuration is called *valid* if it satisfies the requirements of its constrained groups. That is, not all the vertices in the same constrained group have the same colour.

To prove the reducibility of the configurations, we need to check (1) that every valid 3-colouring of the boundary neighbours can be extended to a 3-colouring of the vertices of the configuration, and (2) that the modified graph (obtained by adding the corresponding gadget) does not have any i-cycles, $4 \leq i \leq 8$. Condition (2) can be hand-checked easily by looking at each configuration and the corresponding modified version, and making sure that for every pair of vertices in the original graph that participate in a gadget, the shortest path between them using only the edges of the gadget is not shorter than the shortest such path in the original graph using only the edges that were deleted to construct G'. Condition (1) is checked with a C program.

As we said, the total number of reducible configurations (considering all possible subcases for configurations 4-15 listed in Section 3.3.2) is 77. The first three of these configurations are the ones used in Theorem 3.2.1. Each of the new 74 configurations is listed in Appendix C. Each figure in this list is drawn by hand using a program called *graphwin*, which is one of the standard demo programs included in the package LEDA (Library for Efficient Data types and Algorithms) version 4.1, distributed by Algorithmic Solutions Software GmbH (available at `http://www.algorithmic-solutions.com`). Using this program we can store the adjacency list of the drawn graph in a file and also save the graph as a Postscript figure. Therefore, for each configuration shown in Appendix C, the adjacency list, which is used as input to the program, is generated automatically with the figure. The adjacency lists of all 74 configurations and the information about the constrained group(s) of vertices are put into a single file, with each configuration separated by a blank line. For more detailed information about the format of input see `ftp://ftp.cs.toronto.edu/csrg-technical-reports/458/`.

The program reads the configurations one by one and the corresponding constrained group(s) of vertices. For each configuration the program generates all the possible valid 3-colourings of its boundary neighbours and then checks whether or not each 3-colouring is extendible to a 3-colouring of the uncoloured vertices of the configuration. This check is done using exhaustive search plus a bit of intelligence; the program colours the vertices one at a time and for each uncoloured vertex, the program only considers all possible colours that have not appeared in its neighbourhood. For example, if a vertex already has colours 1 and 2 in its neighbourhood, there is only one colour (i.e. colour 3) that can be assigned to this vertex, and the program does not try colours 1 or 2. If all the valid 3-colourings of the boundary neighbours are extendible, then the configuration is reducible. We didn't attempt to make any other optimizations in the program, since this simple straightforward implementation checks all the reducible configurations very quickly, on a desktop computer, and further optimizations would be at the cost of losing its readability.

Chapter 4

One Further Step on Steinberg's Conjecture

Remark 4.0.1 *The results of this chapter are based on paper [18].*

In this chapter, we tighten the gap between Steinberg's conjecture and the best known result on this problem by improving Theorem 3.1.1. Let $\mathcal{G}_7$ be the class of planar graphs without cycles of size in $\{4, \ldots, 7\}$.

Theorem 4.0.2 *Every graph in* $\mathcal{G}_7$ *is 3-colourable.*

So, we are only two steps away from the conjecture of Steinberg. The proof of Theorem 4.0.2 is more elegant and shorter than that of Theorem 3.1.1. There are just a handful of reducible configurations and the proof is completely hand-checkable.

One important feature of this proof is that it does *not* rely on Theorem 3.1.1. It only uses Example 2.2.4, as the basis of an induction, and the overall proof is much shorter than the proof of Theorem 3.1.1. Consequently, the 3-colouring algorithm that we provide uses only the 3-colouring algorithm of Subsection 2.3 for the base case of a recursion, and therefore, it does not need to check all the configurations described in the previous chapter.

The organization of this chapter is as follows. In the next section, we point out a very simple structure that appears in most of the reducible configurations of the previous chapter. We investigate the required conditions under which we can prove the reducibility of this simple structure. Proving the reducibility of this structure helps us to bring down the total number of reducible configurations, significantly. In Section 4.2 we present the proof of Theorem 4.0.2. This is done by proving a stronger statement, namely Theorem 4.2.1, which in turn implies Theorem 4.0.2. Again, the proof uses the Discharging Method. The reducible configurations are presented and their reducibility is proved in Subsection 4.2.1. Then, in Subsection 4.2.2, we show the unavoidability of these configurations by applying a suitable set of initial charges and discharging rules. Finally, in Section 4.3 we present a 3-colouring algorithm for graphs in $\mathcal{G}_7$, based on the proof of Theorem 4.0.2.

4.1 Some New Ideas

A careful look at the reducible configurations used in the proof of Theorem 3.1.1 suggests that there are very similar patterns that repeat in most of them. So, before trying to prove Theorem 4.0.2, let's see if we can refine our proof ideas, to show the reducibility of most of the configurations considered in the previous chapter, all at once.

A path $v_1v_2v_3v_4$ is called a *tetrad* if $d(v_i) = 3$, $1 \leq i \leq 4$, $\ldots xv_1v_2v_3v_4x' \ldots$ is on the boundary of some face f, and there are triangles tv_1v_2 and $t'v_3v_4$, such that t and t' do not belong to the boundary of f (See Figure 4.1). By this definition, it is easy to see that at least one tetrad appears in most of the configurations used in the previous chapter. So, if we can prove that a tetrad is reducible, that will reduce the number of configurations significantly, and might even help in finding some new reducible configurations.

To do this, let's assume that $G \in \mathcal{G}_7$ is a counter-example with the minimum number of vertices and consider a tetrad in G. Delete v_1, v_2, v_3, and v_4, along with all incident

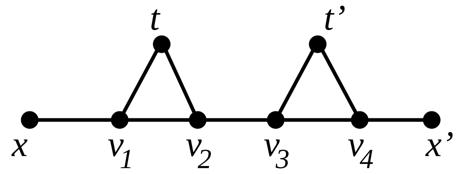


Figure 4.1: A tetrad

edges. Consider a 3-colouring C of this new smaller graph G'. If we could assume that $C(x) = C(t')$ then we could easily extend C to a 3-colouring of G: we first colour v_4 and v_3 (in this order); then since x and v_3 have different colours, it is easy to colour v_1 and v_2. This will show the reducibility of a tetrad. But the assumption that $C(x) = C(t')$ is a crucial point. Can we make this assumption?

One way to make sure that $C(x) = C(t')$ is to identify x with t' in G' before colouring C. But this causes some new problems: this identification may create small cycles (cycles of size in $\{4, \ldots, 7\}$), and therefore we cannot claim that G' is 3-colourable anymore. Can we show that such a cycle cannot exist? If such a small cycle exists in G', then the sequence of vertices of this cycle starting from x, plus $v_1v_2v_3$ forms a cycle in G which separates t from x', i.e. one of t and x' is inside the cycle and the other one outside of it. Now, we have to argue that G cannot have such a cycle, which will be called a *separating* cycle.

Fortunately there is a way to prove something along these lines. Under some assumptions (to be cleared soon), if there exists a separating cycle in G then we can colour the subgraphs of G inside and outside the separating cycle independently, and ensure that their colourings match on this cycle. This shows that such a cycle will be reducible in G. These arguments suggest that if we strengthen our statement (i.e. have a stronger induction hypothesis) we may be able to prove that separating cycles are reducible and from that show that tetrads are reducible, too. This will help us to bring down the number of reducible configurations dramatically.

4.2 Proof of the Main Theorem

Following the arguments of the previous section, in order to prove Theorem 4.0.2, we prove the following stronger theorem:

Theorem 4.2.1 *Consider any connected graph $G \in \mathcal{G}_7$ and let f be any face of G with size in $\{8, \ldots, 11\}$. Every proper 3-colouring of the subgraph induced by the vertices of f can be extended to a proper 3-colouring of G.*

Assuming Theorem 4.2.1, we can easily prove Theorem 4.0.2. Before doing so, we state a couple of definitions. Let C be a cycle of length k whose sequence of vertices is $v_0v_1\ldots v_{k-1}$. An edge between two non-consecutive vertices of this cycle is called a *chord* for C. If a chord is between v_i and v_{i+2}, for some $0 \le i \le k-1$, where the addition is in mod k, then we say this chord *cuts triangle* $v_iv_{i+1}v_{i+2}$ from C, or it is a *triangular* chord.

Proof of Theorem 4.0.2: Suppose that G is a counter-example to Theorem 4.0.2 with the smallest number of vertices. Clearly, G is connected and by Example 2.2.4 it has a cycle C of length in $\{8, 9, 10\}$. By the absence of cycles of length in $\{4, \ldots, 7\}$ in G, C can only have triangular chords, if it has chords at all. Let $e = uv$ be a triangular chord of C, which cuts triangle uwv from C. We call w a *triangular vertex* of C. w cannot be the end-point of any chord of C, otherwise if wx is a chord (which must be a triangular chord) then $\{u, v, w, x\}$ forms a 4-cycle in G (See Figure 4.2). If we remove all the triangular vertices of C, the remaining vertices of C induce a cycle C', which is formed by the chords of C and some of the edges of C. We can find a 3-colouring φ' of C'. Since each triangular vertex of C is adjacent to exactly two coloured vertices of C', we can extend φ' to a 3-colouring φ of all the vertices of C. Now delete the possible chords of C. If we remove the vertices inside C this cycle becomes a face with size in $\{8, 9, 10\}$, and by Theorem 4.2.1, φ can be extended to the vertices of G outside C. Also, if we remove the vertices outside of C from G, by Theorem 4.2.1, φ can be extended to the vertices of G inside C. The union of these two colourings is a 3-colouring of G. ∎

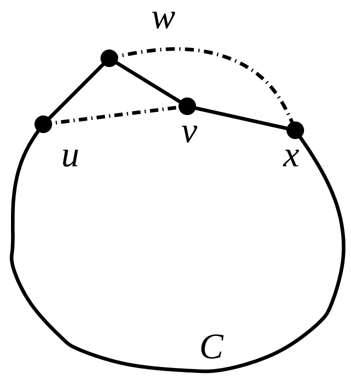


Figure 4.2: A cycle C with triangular chords

In the rest of this section we will give the proof of Theorem 4.2.1. Before starting the proof, we state more notation used in the proof.

Throughout this section, we denote the outside face of an embedded planar graph $G \in \mathcal{G}_7$ by f_0. Any face other than f_0 is *internal*. Also, the vertices of G that do not belong to f_0 are *internal*. We redefine a *bad* vertex to be an *internal* 3-vertex which is incident with a 3-face. Note that this definition is slightly different from that of Chapter 3, as we impose the condition of being an *internal* vertex. Any vertex that is not bad is called a *good* vertex. The set of vertices of G lying inside and outside of a cycle S are denoted by $\mathrm{In}(S)$ and $\mathrm{Out}(S)$, respectively. If $\mathrm{In}(S) \neq \emptyset$ and $\mathrm{Out}(S) \neq \emptyset$, then S is called a *separating* cycle.

4.2.1 Reducible Configurations

In this subsection only, by a *minimum counter-example* we mean a graph $G \in \mathcal{G}_7$ and a 3-colouring φ of the vertices of a face f of G that form a counter-example to Theorem 4.2.1 with the minimum number of vertices. Without loss of generality, we assume that f is the outside face, f_0.

The first two reducible configurations we had in the proofs of Examples 2.2.3 and 2.2.4, and Theorems 3.2.1 and 3.1.1 were ≤ 2-vertices and cut-vertices. First we prove that cut-vertices are reducible for G:

Lemma 4.2.2 *Every minimum counter-example is 2-connected; in particular, it cannot have 1-vertices.*

Proof: Assume that G is a minimum counter-example. If there is a cut-vertex $v \in f_0$, then because $8 \leq |f_0| \leq 11$ and $G \in \mathcal{G}_7$, there is a block B of G containing v which is a single edge or a triangle. In each case it is easy to see that $G - (B - \{v\})$ is a smaller counter-example, contradicting the definition of G.

Now assume that B is a pendant block with cut-vertex $v \notin f_0$. We first extend φ to $G - (B - \{v\})$, then 3-colour B (using the minimality of G), and finally get an extension of φ to G. ∎

For 2-vertices, we cannot prove that they don't exist in a minimum counter-example, but we can show if they exist then they must belong to face f_0. Before proving this, we prove the following lemma which, as we discussed in the previous section, will also be used in the proof of reducibility of tetrads.

Lemma 4.2.3 *A minimum counter-example has no separating cycle of length at most 11.*

Proof: By way of contradiction, assume that G is a minimum counter-example and S is a separating cycle of length at most 11 in G. Because of the minimality of G, we can extend φ to $G - \mathrm{In}(S)$. Let φ_S be the colouring of S in this extension. Then we delete the (possible) chords of S. Thus S becomes a face in $G - \mathrm{Out}(S)$. If $|S| \neq 3$ then $8 \leq |S| \leq 11$, and therefore by the minimality of G, we can extend φ_S to $G - \mathrm{Out}(S)$, thus obtaining a 3-colouring of G.

If $|S| = 3$, either there exists a 3-colouring φ' of $G - \mathrm{Out}(S)$ by Example 2.2.3, or $G - \mathrm{Out}(S)$ has a cycle C of length between 8 and 11. In the latter case, by an argument similar to that of proof of Theorem 4.0.2, we can find a 3-colouring of C and then extend this colouring to a 3-colouring φ' of $G - \mathrm{Out}(\mathrm{S})$, using the minimality of G. Since S is a clique, we can permute the colours in φ' such that φ' on S becomes equal to φ_S. Thus we have 3-colouring of G. ∎

Lemma 4.2.4 *In every minimum counter-example, each 2-vertex belongs to f_0 and no 2-vertex is incident with a 3-face.*

Proof: Let G be a minimum counter-example. If v is an internal 2-vertex of G then we can extend φ to $G - v$ by minimality of G and then colour v.

If v is a 2-vertex in f_0 that belongs to a triangle T then, by Lemma 4.2.3, T is not a separating cycle; so T is a face. Therefore if we remove v from f_0, the size of the boundary of the outer face decreases by exactly one, and all its vertices have a colour in φ. Since $G \in \mathcal{G}_7$, this new face has size in $\{8, 9, 10\}$. Consider this new graph G' obtained by removing v from G. By minimality of G, φ induced on G' can be extended to a 3-colouring of G'. This colouring is also an extension of φ to G, a contradiction. ∎

Using the previous two lemmas we can show that every relatively small cycle in a minimum counter-example has no non-triangular chords.

Lemma 4.2.5 *In a minimum counter-example, no cycle of length at most 13 has a non-triangular chord, and f_0 has no chords at all.*

Proof: Let G be a minimum counter-example. If a cycle C in G has a non-triangular chord it must be divided by this chord into two cycles of length at least 8 each. This implies that $|C| \geq 14$.

For the second part, assume that a chord uv cuts a triangle $T = \{u, v, w\}$ from f_0 in G. Then by Lemma 4.2.3, T is a 3-face, i.e. there are no vertices inside T. This implies that $d(w) = 2$, which contradicts Lemma 4.2.4. ∎

We said in the previous section that one key structure in our proof, that helps to bring down the number of reducible configurations significantly, is a tetrad. Now we are ready to prove that a minimum counter-example cannot have this structure. We call a tetrad $T = v_1v_2v_3v_4$ (as in Figure 4.3(a)) *internal* if v_1, v_2, v_3, and v_4 are internal vertices.

Lemma 4.2.6 *A minimum counter-example cannot have an internal tetrad.*

Proof: By way of contradiction, let G be a minimum counter-example and take an internal tetrad in G (as in Figure 4.3(a)). Note that since G has no cut-vertices, all faces f_1, f_2, f_3, and f_4 are distinct. First delete edges tv_1 and tv_2 from G. Then delete vertex v_4 and contract the following edges: xv_1, v_1v_2, v_2v_3, and v_3t'. Let's call this new graph G^*. Clearly G^* is an embedded planar graph since we removed some vertices and edges from G and then contracted an induced path. In fact, this is a planar embedding of the graph obtained by deleting v_1, v_2, v_3, v_4 and identifying x with t' in G. We will explain later how this might affect the colouring φ if one or both of x or t' are in f_0.

We claim that G^* has no faces of size in $\{4, \ldots, 7\}$: one of the new faces is created from the vertices in $f_1 - \{v_1, v_2, v_3, v_4\}$ and $f_4 - \{v_4\}$, and therefore has size at least $|f_1| - 5 + |f_4| - 2 \geq 9$. The other new face in G^* is created from the vertices in $f_2 - \{v_1\}$ and $f_3 - \{v_2, v_3\}$ (note that x and t' are the same in G^*), and therefore has size at least $|f_2| - 2 + |f_3| - 3 \geq 11$. Hence, if G^* has any cycle of size in $\{4, \ldots, 7\}$, that cycle must be a separating cycle (because it cannot be a face). We now prove that G^* cannot have a separating cycle of size in $\{4, \ldots, 7\}$, either. The only way for G^* to have such a cycle, is to have a path of length in $\{4, \ldots, 7\}$ from x to t in G which does not use any of v_1, v_2, v_3, and v_4. That will create a cycle $S^* = xz_1 \ldots z_k t$, where $3 \leq k \leq 6$. Then $S = xz_1 \ldots z_k tv_3v_2v_1$ separates t from v_4 in G (see Figure 4.3(b)). Indeed, t cannot lie on S by Lemma 4.2.5. But this means that S is a separating cycle in G with size in $\{8, \ldots, 11\}$, which contradicts Lemma 4.2.3.

A loop in G^* corresponds to an edge between x and t' in G. But such an edge, together with $xv_1v_2v_3t$ would create a cycle of size 5 in G. So G^* has no loops. If there are multiple edges in G^* they must be between the unified vertex in G^* (corresponding to x and t) and some other vertex. This means that x and t have some common neighbours in G. But this neighbour, together with $xv_1v_2v_3t$ would form a cycle of size 6 in G. So G^* has no multiple edges, either. Thus $G^* \in \mathcal{G}_7$.

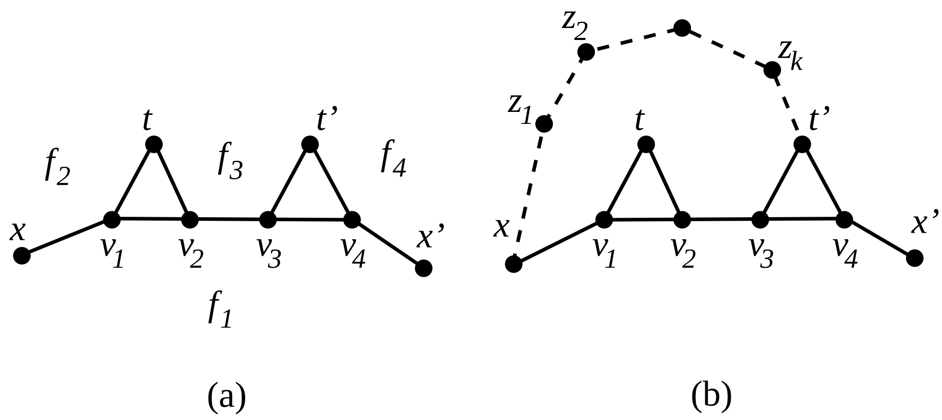


Figure 4.3: (a) A tetrad and (b) the separating cycle

Next recall that any 3-colouring ψ of G^* can be extended to a 3-colouring of G: x and t' each get the colour of the unified vertex. We first colour v_4 and v_3 (in this order); then, since x and v_3 have different colours, it is easy to colour v_1 and v_2. If the colouring φ of f_0 is not damaged by identifying x with t', then by minimality of G, G^* has a 3-colouring that extends φ. This 3-colouring can be extended to a 3-colouring of G which is a contradiction. It follows that while identifying x with t we damaged φ, i.e. we either (a) identified two vertices of f_0 coloured differently, or (b) inserted an edge between two vertices of f_0 coloured the same. For at least one of these two situations to happen, the total of the distances from x to f_0 and from t' to f_0 must be at most 1.

Let $d_1 \dots d_{|f_0|}$ be the sequence of vertices of f_0, with the subscripts increasing in the clockwise order. Suppose d_1 is a vertex of f_0 nearest to x (and possibly equal to x), while d_j is closest to t' (possibly equal to t'). Since $|f_0| \leq 11$, it follows that the boundary of f_0 is split by d_1 and d_j into paths P_1, P_2 one of which, say $P_1 = d_1 \dots d_j$, consists of at most 5 edges. This path, combined with the path $d_j t' v_3 v_2 v_1 d_1$ (for the case that $x = d_1$ and $t' \neq d_j$), or with $d_j v_3 v_2 v_1 x d_1$ (for the case that $x \neq d_1$ and $t' = d_j$), or with $d_j v_3 v_2 v_1 d_1$ (for the case that $x = d_1$ and $t' = d_j$) yields a cycle C of length at most 10 in G. By Lemma 4.2.5, since tv_2 is an edge and $v_2 \in C$, it follows that t cannot belong to C. Recall that by definition of an internal tetrad, $xv_1v_2v_3v_4x'$ is on the boundary of some internal face. Therefore, C separates t from v_4. But this contradicts Lemma 4.2.3. ■

Remark 4.2.7 *By the proof of this lemma, if T is a tetrad as in Figure 4.3(a) in a graph $G \in \mathcal{G}_7$ (which is not necessarily a counter-example) and ψ is a 3-colouring of the vertices of G^* (constructed as in the proof of the lemma), then ψ induced on G (in which x and t' have the same colour as the unified vertex of G^*) can be extended to a 3-colouring of G in constant time.*

Now that we have proved tetrads are reducible, it is not hard to see that most of the reducible configurations we had in the previous chapter are reducible as they have a tetrad. If we were to try to do a proof similar to that of Chapter 3 then many of the reducible configurations that involve 8-faces would have tetrads. So Lemma 4.2.6 can be used to eliminate most of them. As a result we only have to introduce two new reducible configurations. We define them below and show that they are reducible.

Let f be an 8-face with boundary $v_1, \ldots, v_8$ (in counter-clockwise order), where $v_1, v_2, v_3, v_5, v_6, v_7$ are bad, while v_4 and v_8 are internal good vertices. Assume that $v_2v_3t_{23}$, $v_5v_6t_{56}$, $v_1v_8t_{18}$, and $v_7v_8t_{78}$ are 3-faces adjacent to f (see Figure 4.4(a)). So $d(v_8) = 4$ and $d(v_4) \geq 3$. Then f is called an *M-face.*

Lemma 4.2.8 *A minimum counter-example cannot have an M-face.*

Proof: Assume that G is a minimum counter-example with an M-face f as in Figure 4.4(a). We obtain G^* from G by deleting all the bad vertices of f and identifying v_4 with v_8. As in the proof of Lemma 4.2.6, it is easy to check that G^* does not have a face of size in $\{4, \ldots, 7\}$, and it cannot have a separating cycle of size in $\{4, \ldots, 7\}$, or else G has a separating cycle of size in $\{8, \ldots, 11\}$ containing $t_{18}v_1v_2v_3v_4$ (separating t_{23} from v_8), or a cycle of size in $\{8, \ldots, 11\}$ containing $t_{78}v_7v_6v_5v_4$ (separating t_{56} from v_8), thus contradicting Lemma 4.2.3 (see Figure 4.4(b)). Also, G^* has neither loops nor multiple

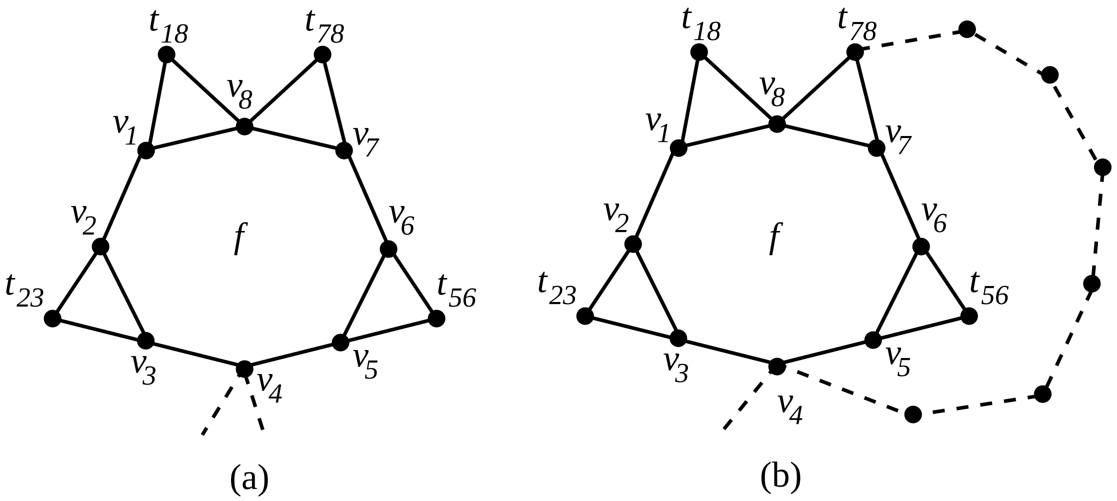


Figure 4.4: (a)An M-face f, (b) a possible separating cycle

edges, or else G would have a cycle of size in $\{4, 5, 6\}$ containing v_5, v_6, v_7. Therefore, G^* has no cycles of size in $\{4, \ldots, 7\}$, i.e. $G^* \in \mathcal{G}_7$.

The same arguments as in the last two paragraphs of the proof of Lemma 4.2.6 show that the colouring φ of f_0 is not damaged by identifying v_4 with v_8, as otherwise G would have a cycle of size at most 11 through $v_4v_5v_6v_7t_{78}$ (or $v_4v_3v_2v_1t_{18}$) which separates t_{56} from v_8 (or t_{23} from v_8), thus contradicting Lemma 4.2.3.

Since G^* is smaller than G, φ can be extended to a 3-colouring ψ of G^*. We will show that ψ can be extended to a 3-colouring of G. Consider ψ induced on G and give v_4 and v_8 the same colour as the unified vertex in G^*. First colour v_1 and v_7. Since $\psi(v_4) \neq \psi(v_1)$ and $\psi(v_4) \neq \psi(v_7)$, we can easily extend this colouring to v_2, v_3, v_5, and v_6. ■

Remark 4.2.9 *By the proof of this lemma, if f is an M-face as in Figure 4.4(a) in a graph $G \in \mathcal{G}_7$ (which is not necessarily a counter-example) and ψ is a 3-colouring of the vertices of G^* (constructed as in the proof), then this colouring induced on G can be extended to a 3-colouring of G in constant time.*

The other structure we define is very similar to the previous one. Let f be an 8-face with boundary vertices $v_1, \ldots, v_8$ (in counter-clockwise order), where $v_1, \ldots, v_4$ and v_6, v_7

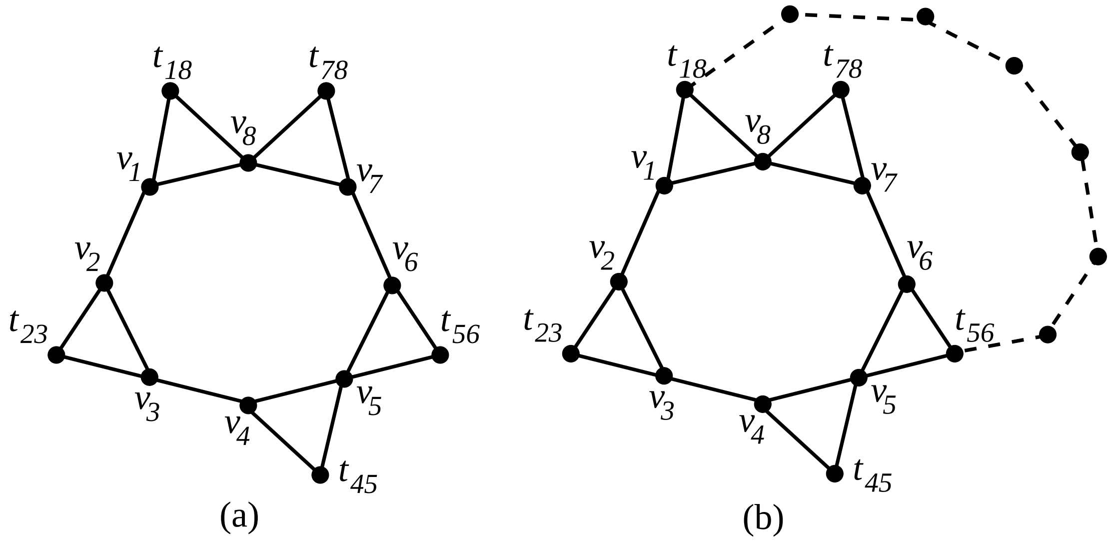


Figure 4.5: (a) An MM-face, (b)a possible separating cycle

are bad vertices, while v_5 and v_8 are internal 4-vertices. Assume that $v_2v_3t_{23}$, $v_4v_5t_{45}$, $v_5v_6t_{56}$, $v_7v_8t_{78}$, and $v_8v_1t_{18}$ are 3-faces adjacent to f (see Figure 4.5(a)). Then f is called an *MM-face.*

Lemma 4.2.10 *A minimum counter-example cannot have an MM-face.*

Proof: By way of contradiction, let G be a minimum counter-example and f an MM-face of G as in Figure 4.5(a). We obtain G^* from G by deleting $v_1, \ldots, v_8$ and identifying t_{18} with t_{56}. As in the previous two lemmas, it is easy to check that $G^* \in \mathcal{G}_7$. Otherwise there is a cycle of size at most 11 in G through $t_{56}v_6v_7v_8t_{18}$ (see Figure 4.5(b)), which separates t_{78} from v_5, contradicting Lemma 4.2.3. Also, as in the previous two lemmas, the colouring φ of f_0 is not damaged by this identification, or else there is a cycle of size at most 11 through $t_{56}v_6v_7v_8t_{18}$ separating v_5 and t_{78}, which contradicts Lemma 4.2.3.

Now we show that every 3-colouring ψ of G^* can be extended to a 3-colouring of G. Let ψ be an arbitrary 3-colouring of G^* and consider ψ induced on G, with t_{18} and t_{56} having the same colour as the unified vertex of G^*. Without loss of generality, assume that $\psi(t_{18}) = \psi(t_{56}) = 1$. If $\psi(t_{45}) \neq 1$, we first colour v_5, v_4, and v_6, (in this order); then, using an argument as in the proof of Lemma 4.2.6, we can colour v_8

and v_7, then v_1, and finally v_2 and v_3, as $\psi(v_4) = 1 \neq \psi(v_1)$. If $\psi(t_{45}) = 1$, we set $1 \neq \psi(v_8) = \psi(v_6) = \psi(v_4) \neq \psi(t_{78})$, then colour v_1, v_5, v_7 (in this order), and finally v_2 and v_3. ■

Remark 4.2.11 *By the proof of this lemma, if f is an MM-face as in Figure 4.5(a) in a graph $G \in \mathcal{G}_7$ (which is not necessarily a counter-example) and ψ is a 3-colouring of the vertices of G^* (constructed as in the proof), then this colouring induced on G can be extended to a 3-colouring of G in constant time.*

In summary, here is the list of configurations that are proved to be reducible in Lemmas 4.2.2-4.2.10:

1. A cut-vertex
2. A separating cycle of length at most 11
3. An internal 2-vertex
4. A 2-vertex in f_0 incident with a 3-face
5. A chord in f_0
6. An internal tetrad
7. An M-face
8. An MM-face

In the next subsection we prove that this set of reducible configurations is unavoidable, using the Discharging Method.

4.2.2 Discharging Rules

Let G be an arbitrary connected graph in $\mathcal{G}_7$ given with a proper 3-colouring of the vertices of one of its faces f_0, with $8 \leq |f_0| \leq 11$. Consider an embedding of G in which

f_0 is the outside face. We use the Discharging Method to show that G must have one of the reducible configurations listed in the previous subsection.

The initial charges we apply are very similar to the ones we have seen in Chapters 2 and 3. To each vertex v we assign $d(v) - 6$ units of charge and to each face $f \neq f_0$ we assign $2|f| - 6$ units. The only difference is that we assign $2|f_0| + 5.5$ units of charge to f_0. We need to do this because of the possible presence of 2-vertices on f_0. Using Euler's formula, the total charge is

$$\sum_{v \in V} (d(v) - 6) + \sum_{f \neq f_0} (2|f| - 6) + 2|f_0| + 5.5 = -\frac{1}{2}.$$

In the discharging phase we move charges from faces to vertices and show that after this phase every vertex and face has non-negative charge (and therefore the total charge is non-negative), unless G has one of the reducible configurations listed in the previous subsection. Of course, if G has a reducible configuration then G cannot be a minimum counter-example. This shows that there is no minimum counter-example to Theorem 4.2.1.

It is easy to see that by this set of initial charges, the only elements with negative initial charge are 2- to 5-vertices. First assume that v is a 2-vertex incident with two faces f and f'. These two faces must be distinct or else v is a cut-vertex, which is reducible configuration 1. Since the initial charge of v is -4, f and f' must send 4 units of charge in total to v. One of these two faces, say f, is the outside face, i.e. $f = f_0$, or else we have reducible configuration 3. Because f_0 has larger charge/size ratio with respect to the other faces, it seems better to send more charge from f_0 to v than from the internal face f'. So, instead of sending 2 units of charge from each of f_0 and f' to v, we send $\frac{5}{2}$ units of charge from f_0 and $\frac{3}{2}$ from f' to v. In fact it is not hard to see that f_0 can afford to send $\frac{5}{2}$ units of charge to every vertex $v \in f_0$: the initial charge of f_0 is $2|f_0| + 5.5$ and if it sends $\frac{5}{2}|f_0|$ it is left with $5.5 - \frac{|f_0|}{2}$ units of charge. Since $8 \leq |f_0| \leq 11$, the final charge of f_0 will be non-negative. So we introduce the following rules:

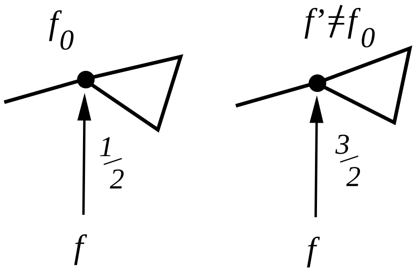


Figure 4.6: Discharging rule R3

R1: f_0 sends $\frac{5}{2}$ to each of its vertices.

R2: Every internal non-triangular face sends $\frac{3}{2}$ units to its 2-vertices.

If a 2-vertex v does not belong to f_0, then G has reducible configuration 3 and we are done. Otherwise, every 2-vertex v belongs to f_0 and receives $\frac{5}{2}$ from f_0 by R1. Also, the other face incident with v is a non-triangular face and sends $\frac{3}{2}$ to v by R2, or else G has a 2-vertex in f_0 incident with a triangular face, which is reducible configuration 4. Therefore, these two rules ensure that either every 2-vertex v has non-negative charge, or G has reducible configuration 3 or 4.

The first discharging rule in the proofs of Theorems 3.2.1 and 3.1.1 was to send $\frac{3}{2}$ from "large" non-triangular faces to each of their bad vertices. Here we keep this rule, with slight modifications. If $v \in f_0$ is a 3-vertex incident with a triangle, it receives $\frac{5}{2}$ from f_0 by R1, and it only requires $\frac{1}{2}$ from the internal non-triangular face. Note that by the definition of bad in this chapter, v is not bad (because it is not internal). Here is the new rule:

R3: Every internal non-triangular face f sends $\frac{3}{2}$ units to each of its bad vertices and $\frac{1}{2}$ to every 3-vertex in its boundary that also belongs to f_0 and is incident with one 3-face (see Figure 4.6).

Recall the definition of a simple vertex from the previous chapter: a 3-vertex not incident with any triangles. These vertices have initial charge -3 and so require 3 units

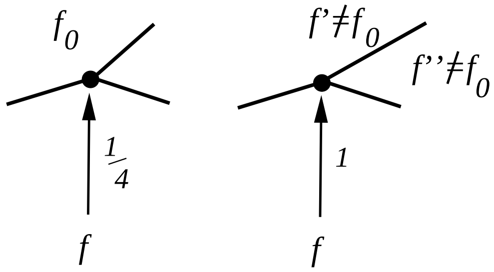


Figure 4.7: Discharging rule R4

of charge. If such a vertex is in f_0 it gets $\frac{5}{2}$ of charge from f_0 and it only needs $\frac{1}{4}$ units from each of the other (internal) faces it is incident with. Otherwise, each of the faces must send 1 unit to it. So:

> **R4:** Every internal non-triangular face f sends 1 unit to each of its internal simple vertices and $\frac{1}{4}$ to each of its simple vertices that also belongs to f_0. (see Figure 4.7)

Rules R1-R4 ensure that every 3-vertex which is not a cut-vertex (reducible configuration 1) has non-negative final charge: if v is a 3-vertex and is in f_0 it receives $\frac{5}{2}$ from f_0 by R1 and $\frac{1}{2}$ by rules R3 or R4 from the other non-triangular face incident with it, depending on whether it is incident with a triangle or is simple. If $v \notin f_0$ then if it is bad it receives $2 \times \frac{3}{2}$ by R3 and if it is simple it receives 3×1 by R4.

The only remaining vertices are 4- and 5-vertices. If a ≥ 4-vertex v belongs to f_0 it receives $\frac{5}{2}$ from f_0 and so has positive charge. Thus we only need to deal with *internal* 4- and 5-vertices.

Recall from Chapter 3 that a type 0, type 1, or a type 2 vertex is a 4-vertex incident with 0, 1, or 2 triangles, respectively. Every 4-vertex is one of these types. Every 4-vertex v is incident with 4 distinct faces, otherwise v is a cut-vertex which is reducible configuration 1. If v is an internal type 0 vertex it is enough to send $\frac{1}{2}$ units to it from each of its faces. If v is a type 2 vertex, it needs to get 1 unit from each of its non-triangular faces to have non-negative charge. Finally, if v is a type 1 vertex, we can send $\frac{1}{2}$ from

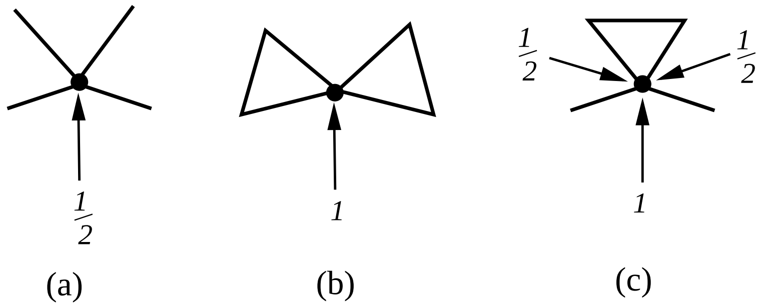


Figure 4.8: Discharging rule R5

each of the non-triangular faces incident with v that share an edge with the triangular face, and 1 unit from the other face to v. We combine these in the following rule (see Figure 4.8):

R5: Every internal non-triangular face f sends:

(a) $\frac{1}{2}$ to each of its internal type 0 vertices,

(b) 1 to each of its internal type 2 vertices,

(c) $\frac{1}{2}$ to every internal type 1 vertex v in its boundary if the triangle incident with v shares an edge with f,

(d) 1 to every internal type 1 vertex v in its boundary if the triangle incident with v does not share an edge with f.

Let's assume v is an internal 4-vertex. If it is type 0, type 2, or type 1 it receives $4 \times \frac{1}{2}$, or 2×1, or $2 \times \frac{1}{2} + 1$ by R5 parts (a), or (b), or (c) and (d), respectively. So by rules R1 and R5 every 4-vertex either has non-negative charge after the discharging phase, or is a cut-vertex (reducible configuration 1).

The only remaining vertices are internal 5-vertices. Let v be an internal 5-vertex incident with 5 faces. All these faces are distinct, otherwise v is a cut-vertex which is reducible configuration 1. By absence of 4-cycles, v is incident with at least three non-triangular faces and it is enough to send $\frac{1}{2}$ from two of them to v. So we add the following discharging rule to our set:

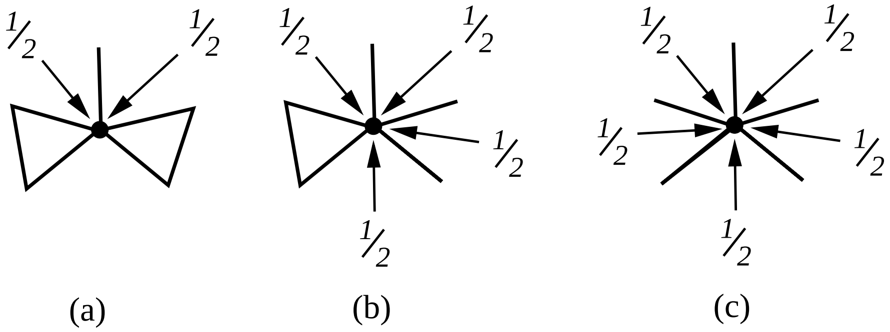


Figure 4.9: Discharging rule R6

R6: Every internal non-triangular face f sends $\frac{1}{2}$ to each internal 5-vertex v in its boundary if v is not incident with two edges of f that each belong to a triangular face adjacent to f (see Figure 4.9).

If v is an internal 5-vertex then it is incident with at least three non-triangular faces. If it is incident with at least 4 non-triangular faces then each of them sends $\frac{1}{2}$ to v by R6, for a total of at least 2. If v is incident with two triangles then two of the non-triangular faces send $\frac{1}{2}$ each by R6, for a total of 1.

Therefore, by these discharging rules:

Lemma 4.2.12 *Every vertex v has non-negative charge, unless it is reducible configuration 1, 3 or 4.*

Now we prove that every face has non-negative charge, or else G has a reducible configuration. Since R1 is the only rule by which f_0 sends charge, by the arguments given before R1:

Lemma 4.2.13 *f_0 has non-negative charge after the discharging phase.*

Finally, we show that every internal face f either has non-negative charge, or has a reducible configuration.

Lemma 4.2.14 *Every face $f \neq f_0$ has non-negative final charge, unless it has reducible configuration 3, 6, 7 or 8.*

Proof: If $|f| = 3$ then its initial charge is 0 and it does not lose any charge in the discharging phase.

Suppose $|f| \geq 12$. As f sends to each incident vertex at most $\frac{3}{2}$ by R2-R6, its final charge is $2|f| - 6 - \frac{3}{2}|f| \geq 0$.

The only remaining cases are when $8 \leq |f| \leq 11$. Assume that f is an internal face with $|f| \geq 8$, which is incident with a 2-vertex v. If $v \notin f_0$ then G has reducible configuration 3. Otherwise f is incident with two $\geq$3-vertices of f_0, namely the ends of a maximal path of 2-vertices on the boundary of f. These vertices get at most $\frac{1}{2}$ from f by R3 and R4, and therefore, the final charge of f is at least $2|f| - 6 - (|f| - 2) \times \frac{3}{2} - 2 \times \frac{1}{2} \geq \frac{|f|}{2} - 4 \geq 0$. Thus, from now on, we may assume that f is not incident with any 2-vertices.

Also, observe that f sends $\frac{3}{2}$ to each of its bad vertices by R3 and at most 1 to its good vertices by rules R4 to R6 (note that since we have assumed that f has no 2-vertices, R3-R6 are the only rules that apply to f). We will use this fact frequently in our arguments without referring to it explicitly.

Suppose $|f| = 11$. By parity, f can have at most 10 bad vertices and sends at most $10 \times \frac{3}{2}$ to them by R3, plus at most 1 to its good vertex by R4, R5, or R6. So, its final charge is at least $22 - 6 - 10 \times \frac{3}{2} - 1 = 0$

Now suppose $|f| = 10$. If f sends to at least two incident vertices at most 1 each, it sends at most $8 \times \frac{3}{2}$ to its other vertices and we are done, as its final charge is at least $20 - 6 - 8 \times \frac{3}{2} - 2 = 0$. The only danger comes from f being incident with at least 9 bad vertices. But clearly every 5 consecutive bad vertices on the boundary of f include a tetrad, which is reducible configuration 6.

Next suppose $|f| = 9$. If f sends to at least three incident vertices at most 1 each, or sends at most $\frac{1}{2}$ to one vertex and 1 to another vertex, we are done, as its final charge is at least $18 - 6 - 6 \times \frac{3}{2} - 3 = 0$ or $18 - 6 - 7 \times \frac{3}{2} - 1 - \frac{1}{2} = 0$, respectively. If f has 8 bad vertices it will certainly form a tetrad, which is reducible configuration 6. So, there are at most 7 bad vertices and the other two must be internal vertices and take 1 from

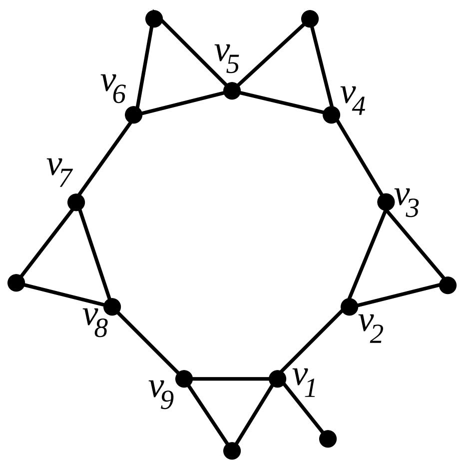


Figure 4.10: A 9-face as in the proof of Lemma 4.2.14

f, each. So, the good vertices are ≤ 4-vertices. Clearly those 7 bad vertices must be split by the two good vertices as 4+3, otherwise they form a tetrad, which is reducible configuration 6. Furthermore, the quadruple should fail to be a tetrad, or else we are done. It is not difficult to check that the only structure that f may have is as in Figure 4.10. But in this case, one of the good vertices (v_1 in the figure) takes $\frac{1}{2}$ from f by R5(c) and the other good vertex, v_5, gets only 1 by R5(b). Therefore the final charge of f is at least $18 - 6 - 7 \times \frac{3}{2} - 1 - \frac{1}{2} = 0$.

Finally, assume $|f| = 8$. This case is more complicated and requires some care to analyze. If there are at most 4 bad vertices in f, or if f sends at most $\frac{1}{2}$ to at least two vertices, then we are done, as its final charge is at least $16 - 6 - 4 \times \frac{3}{2} - 4 = 0$ or $16 - 6 - 6 \times \frac{3}{2} - 2 \times \frac{1}{2} = 0$, respectively. So we may assume that f has at least 5 bad vertices (which by definition are all internal). We prove that the other three vertices of f are also internal.

First suppose that exactly one vertex v of f belongs to f_0. Then f cannot share an edge with f_0 (or else it will share at least two vertices with f_0). Thus $d(v) \geq 4$ and so f sends nothing to v by any rules (rules R2 and R3 only apply to 3-vertices and rules R4, R5 and R6 only apply to the *internal* ≥ 4-vertices). If the other 7 vertices of f are all bad, G has a tetrad which is reducible configuration 6. Otherwise, f has at most 6 bad vertices and sends at most 1 to its other good vertex. So f has at least $16 - 6 - 6 \times \frac{3}{2} - 1 = 0$ final charge. Now suppose that at least two vertices of f belong to f_0. Then, since f sends at most $\frac{1}{2}$ to each of them by R3 or R4, f has non-negative charge as discussed in the previous paragraph. So all the vertices of f must be internal.

If f is incident with at least 7 bad vertices (and so with at most one good vertex), G has a tetrad and we are done. The only other case we have to consider is when f is incident with exactly 6 or exactly 5 bad vertices. Let $v_1 \dots v_8$ be the sequence of vertices of f in clockwise order.

Case 1. f has precisely 5 bad vertices.

If at least one good vertex of f gets at most $\frac{1}{2}$ from f, since the other two good vertices get at most 1 from f each, we are done, as the final charge of f is at least $16 - 6 - 5 \times \frac{3}{2} - 2 \times 1 - \frac{1}{2} = 0$. So suppose that each of these three good vertices takes 1 unit of charge from f. It follows by R4 and R5 that all of them are internal ≤ 4-vertices, and each is either (i) simple, (ii) type 2, or (iii) a type 1 vertex which is is incident with a triangle that does not share any edges with f. However, this is impossible by parity: the number of bad vertices should be even or else f should contain a type 1 vertex which is incident with a 3-face that shares an edge with f (i.e. is adjacent to f).

Case 2. f has precisely 6 bad vertices.

These 6 bad vertices must be split by the two good vertices as 4+2 or 3+3, since each path of 5 bad vertices contains a tetrad, and tetrads are reducible. We consider each of these two subcases separately:

Subcase 2.1: 4+2

Assume that the group of 4 bad vertices is $v_1, \dots, v_4$ and the other two bad vertices are v_6, v_7, with v_5 and v_8 being good. In order to not form a tetrad, v_1 and v_4 should form triangles with the good vertices v_8 and v_5, respectively. Let's call an edge incident with a 3-face a *triangular* edge. If the edge v_6v_7 is triangular, then both v_5 and v_8 get at most $\frac{1}{2}$ from f by R5(c) or R6, and we are done, as the charge of f is at least

$16 - 6 - 6 \times \frac{3}{2} - 2 \times \frac{1}{2} = 0$.

The only alternative is that both v_5v_6 and v_7v_8 are triangular. Observe that $d(v_5) \geq 4$ and $d(v_8) \geq 4$. If $d(v_5) \geq 5$ then since it is incident with two triangular edges (v_5v_6 and v_4v_5) rule R6 does not apply and f sends nothing to v_5. By a similar argument, if $d(v_8) \geq 5$ then f sends nothing to v_8. Therefore, if $d(v_5) \geq 5$ or $d(v_8) \geq 5$ then we are done as the final charge of f is at least $16 - 6 - 6 \times \frac{3}{2} - 1 = 0$. Thus, the only remaining case to consider is when both v_5 and v_8 are internal 4-vertices and furthermore, we have 3-faces $v_1v_8t_{18}$, $v_2v_3t_{23}$, $v_4v_5t_{45}$, $v_5v_6t_{56}$, and $v_7v_8t_{78}$ as in Figure 4.5(a). But this is an MM-face, i.e. reducible configuration 8.

Subcase 2.2: 3+3

Let $v_1, \ldots, v_8$ be the sequence of vertices of f in clockwise order, with v_4 and v_8 being the good vertices. Without loss of generality assume that v_1v_2 is a triangular edge. So v_3v_4 is also triangular.

If v_5v_6 is triangular then v_7v_8 must be triangular and therefore, v_4 and v_8 take at most $\frac{1}{2}$ from f by R5(c) or R6 and the charge of f is at least $16 - 6 - 6 \times \frac{3}{2} - 2 \times \frac{1}{2} = 0$.

If v_5v_6 is not triangular then v_5v_4 and v_6v_7 are triangular. If $d(v_4) \geq 5$ then f sends nothing to v_4 and therefore its final charge is at least $16 - 6 - 6 \times \frac{3}{2} - 1 = 0$. If $d(v_4) = 4$ then then f is an M-face (as in Figure 4.4(a)), i.e. reducible configuration 7. So we are done. ■

So by Lemmas 4.2.12 and 4.2.14 all the vertices and faces have non-negative final charge, or else G has a reducible configuration. Thus there is no minimum counter-example and so no counter-example at all to Theorem 4.2.1.

4.3 A 3-Colouring Algorithm for Planar Graphs Without 4- to 7-Cycles

In this section we provide an algorithm for Theorem 4.0.2 that given an embedded graph $G \in \mathcal{G}_7$ as input, produces a 3-colouring of G. We assume that the input to the algorithm is connected. For disconnected graphs it is enough to colour each connected component independently. The algorithm consist of two main procedures.

Procedure 1: This procedure takes as input an embedded connected graph $G \in \mathcal{G}_7$ and produces a 3-colouring of G. In the first part of this procedure we apply the algorithm described in Subsection 2.3.1 to G. This will either produce a 3-colouring of G or give a cycle C of size in $\{8, 9, 10\}$ in G. If we find a 3-colouring of G then the procedure returns this 3-colouring and terminates.

Otherwise, let C be the cycle of G that the procedure has found. The same arguments as in the proof of Theorem 4.0.2 show that C can only have triangular chords (because $G \in \mathcal{G}_7$) and that we can find a 3-colouring φ of C. Remove the (possible) chords from C and consider φ on C and the graphs $G_1 = G - \mathrm{In}(C)$ and $G_2 = G - \mathrm{Out}(C)$. Therefore, each of G_1 and G_2 is a connected graph in $\mathcal{G}_7$ along with a 3-colouring of one of its faces (the one whose boundary is C), which has size in $\{8, 9, 10\}$. Then we call Procedure 2 on each of these graphs along with colouring φ, independently. This will produce a 3-colouring for each of G_1 and G_2 that are extensions of φ. The union of these 3-colourings is a 3-colouring of G.

Procedure 2: This procedure takes as input an embedded connected graph $G \in \mathcal{G}_7$ together with a 3-colouring φ of a face f_0 of size in $\{8, \ldots, 11\}$ of G and produces a 3-colouring of G. In fact, this procedure corresponds to Theorem 4.2.1.

We assume that f_0 is the outside face of G. At each iteration of this procedure, we apply the initial charges and the discharging rules, as described in Subsection 4.2.2. Since the total charge is negative, after the discharging phase there must be either a vertex v

or a face $f \neq f_0$ with negative charge (note that by Lemma 4.2.13 f_0 has non-negative charge):

1. *A vertex v with negative charge:* By Lemma 4.2.12, v must be one of configurations 1, 3, or 4. We consider each case separately.

   (a) First assume that v is a cut-vertex. If $v \in f_0$ then, because $G \in \mathcal{G}_7$, there is a block B of G containing v which is a single edge or a triangle. In each case we get an extension of φ to $G-(B-\{v\})$, by calling Procedure 2 recursively. If v is an internal cut-vertex with a pendant block B then we get an extension of φ to $G-(B-\{v\})$, by calling Procedure 2 recursively. Then we run Procedure 1 on B to obtain a 3-colouring of B. This 3-colouring, after possibly permuting the colours, together with the extension of φ to $G-(B-\{v\})$ yield a 3-colouring of G.

   (b) Next assume that v is an internal 2-vertex. We call Procedure 2 or Procedure 1 on each of the at most two connected components of $G - v$, depending on whether the component contains f_0 (and the colouring φ) or not. If Procedure 2 is called on a connected component, say G_2, it returns a 3-colouring of G_2, which is an extension of φ. If Procedure 1 is called on a connected component, say G_1, then it returns a 3-colouring of G_1. The union of these two 3-colourings yields a 3-colouring of $G - v$. We can extend this 3-colouring to v in constant time.

   (c) Finally, assume that v is a 2-vertex of f_0 incident with a triangle T. If T is a face then we call Procedure 2 on $G - v$ and the colouring φ induced on $f_0 - v$, to obtain a 3-colouring of G. This colouring, together with the colour of v induced by φ, yields a 3-colouring of G. If T is a separating cycle, then we call Procedure 2 with $G - \mathrm{In}(T)$ and colouring φ, to obtain a 3-colouring of $G - \mathrm{In}(T)$. Then we call Procedure 1 on graph $G - \mathrm{Out}(T)$. This will produce a 3-colouring of $G - \mathrm{Out}(T)$. The union of 3-colourings of $G - \mathrm{In}(T)$ and $G - \mathrm{Out}(T)$, after possibly permuting the colours in the colouring of $G - \mathrm{Out}(\mathrm{T})$, yields a 3-colouring of G.

2. *A face $f \neq f_0$ with negative charge:* By Lemma 4.2.14 f must have one of configurations 3, 6, 7, or 8. We consider each case separately.

   (a) If f has an internal 2-vertex we do as explained in case 1(b).

   (b) Suppose f has a tetrad as in Figure 4.3. By a Breadth First Search (BFS) starting at vertex x, we can easily check whether there exists a path $x, z_1, \ldots, z_k, t$, $3 \leq k \leq 6$, with all z_i's different from v_1, v_2, v_3, v_4.

      i. If such a path exists, we have a separating cycle C of size in $\{8, \ldots, 11\}$ in G. In this case we call Procedure 2 on $G - \mathrm{In}(C)$ to obtain a 3-colouring of it. Let φ_C be the colouring of C in this 3-colouring. Then we delete the possible chords from C and call Procedure 2 with $G - \mathrm{Out}(C)$ and φ_C. We obtain a 3-colouring of $G - \mathrm{Out}(C)$. The union of these two 3-colourings yields a 3-colouring of G

      ii. If such a path does not exist then we remove v_1, v_2, v_3, v_4 and identify x with t' as in the proof of Lemma 4.2.6. Let this new graph be G^*. We call Procedure 2 on G^* together with φ. This gives a 3-colouring of G^*. By Remark 4.2.7 we can extend this colouring to a 3-colouring of G in constant time.

   (c) Next, suppose that f is an M-face as in Figure 4.4. By a BFS starting from v_4 we check whether there exists a path of length in $\{4, \ldots, 7\}$ in G connecting v_4 to t_{78} (or v_4 to t_{18}) which does not use any edge of f.

      i. If the path exists then this path, together with $v_4v_5v_6v_7t_{78}$ (or with $v_4v_3v_2v_1t_{18}$) forms a separating cycle C of length in $\{8, \ldots, 11\}$ in G. We continue as in case 2(b)i explained above.

ii. If the path does not exist then we remove all the bad vertices of f and identify v_4 with v_8 to obtain graph G^*. We call Procedure 2 on G^* together with φ to get a 3-colouring of G^*. By Remark 4.2.9 this 3-colouring can be extended to G in constant time.

(d) Finally, assume that f is an MM-face as in Figure 4.5. By a BFS starting from t_{18} we check whether there exists a path of length in $\{4, \ldots, 7\}$ between t_{18} and t_{56} that does not use any edge of f.

i. If such a path exists then this path, together with $t_{18}v_8v_7v_6t_{56}$ forms a separating cycle C of length in $\{8, \ldots, 11\}$. We continue as in case 2(b)i explained above.

ii. If the path does not exist then we remove all $v_1, \ldots, v_8$ from f and identify t_{18} with t_{56} to obtain graph G^*. We call Procedure 2 on G^* together with φ to get a 3-colouring of G^*. By Remark 4.2.11 this 3-colouring can be extended to G in constant time.

The main procedure of the algorithm starts by calling Procedure 1. In each procedure if the graph has only one vertex then the procedure immediately returns the trivial colouring of the input graph.

4.3.1 Analysis of the Algorithm

For a graph G, let $n = |V| + |E|$ denote the size of G. Let $T_1(n)$ and $T_2(n)$ be the worst case running time of Procedure 1 and Procedure 2 on an input graph of size n, respectively. Our goal is to show that $T_1(n), T_2(n) \in O(n^3)$. We do this by proving that there are constants $\alpha, \beta_1, \beta_2 > 0$, such that for all values of $n \geq 1$: $T_1(n) \leq \alpha n^3 + \beta_1 n^2$ and $T_2(n) \leq \alpha n^3 + \beta_2 n^2$. Both of the inequalities are trivial for small values of n. Assume that $T_1(i) \leq \alpha i^3 + \beta_1 i^2$ and $T_2(i) \leq \alpha i^3 + \beta_2 i^2$ for $1 \leq i < n$ and suppose that the input graph has size n.

First consider Procedure 1. The part where we run the algorithm of Subsection 2.3.1 takes $O(n^2)$ time. If a 3-colouring is found the procedure terminates. Otherwise, the procedure has found a cycle C. Removing the triangular vertices of C (as in the proof of Theorem 4.0.2) and finding a 3-colouring of cycle C can be done in linear time. Then we should remove the possible chords of C, which again can be done in linear time. If we remove this cycle from the graph we can easily find $G_1 = G - \text{In}(C)$ and $G_2 = G - \text{Out}(C)$ in linear time. Then we make recursive calls to Procedure 2 on G_1 and G_2, which take $T_2(n_1)$ and $T_2(n_2)$ time, if n_1 and n_2 are the sizes of G_1 and G_2, respectively. Note that $n_1, n_2 \geq 8$ and $n_1 + n_2 \leq n + 11$, since the size of C is in $\{8, \ldots, 11\}$. Thus $T_1(n) \leq \gamma n^2 + T_2(n_1) + T_2(n_2) \leq \gamma n^2 + \alpha(n_1^3 + n_2^3) + \beta_2(n_1^2 + n_2^2)$, for some constant $\gamma > 0$. This is maximized when $n_1 = n$ and $n_2 = 11$. So $T_1(n) \leq \gamma n^2 + \alpha(n^3 + 11^3) + \beta_2(n^2 + 11^2) \leq \alpha n^3 + \beta_1 n^2$, if $\beta_1 > \beta_2 + \gamma$.

Now consider Procedure 2. Applying the initial charges takes $O(n)$. Since only faces send charge during the discharging phase and for each face f it takes at most $O(|f|)$ time to do the discharging, it takes at most $O(\sum_{f \in F} |f|)$ time, which is in $O(n)$, to apply the discharging rules. Finding an element with negative charge also takes linear time. Now we analyze each step of this procedure:

1. *A vertex v with negative charge:*

   (a) Checking if a vertex is a cut-vertex can be done in linear time. If v is a cut-vertex and in f_0 then we only make a recursive call to Procedure 2 on a graph with size $n' \leq n - 1$. So for some constant $\gamma > 0$: $T_2(n) \leq \gamma n + T_2(n') \leq \gamma n + \alpha n'^3 + \beta_2 n'^2 \leq \alpha n^3 + \beta_2 n^2$.

   If v is an internal cut-vertex we make a call to Procedure 2 on a graph of size n_2 and a call to Procedure 1 on a graph of size n_1, with $n_1 + n_2 = n + 1$ and $n_1, n_2 \geq 2$. This takes at most $T_2(n_2) + T_1(n_1) \leq \alpha(n_2^3 + n_1^3) + \beta_1 n_1^2 + \beta_2 n_2^2$ time, which is maximized when $n_1 = n - 1$ and $n_2 = 2$, since $\beta_1 > \beta_2$. After this step we may have to permute the colours in one of the colourings obtained, which takes linear time. Therefore, $T_2(n) \leq \gamma n + \alpha((n-1)^3 + 2^3) + \beta_1(n-1)^2 + \beta_2 2^2$, for some constant $\gamma > 0$. This implies that $T_2(n) \leq \alpha n^3 + \beta_2 n^2$, if α is large enough with respect to β_1 and β_2.

(b) Checking if v is a 2-vertex takes constant time. If v is an internal 2-vertex then we call Procedure 1 or Procedure 2 on each of the at most two connected components of $G - v$. Suppose that Procedure 2 is called on a connected component of size n_2 and Procedure 1 is called on a connected component of size n_1, with $n_1 + n_2 = n - 1$ and $n_1, n_2 \geq 0$. Then we take the union of these two colourings and extend it to v in constant time. So $T_2(n) \leq T_1(n_1) + T_2(n_2) + \gamma n \leq \alpha(n_1^3 + n_2^3) + \beta_1 n_1^2 + \beta_2 n_2^2 + \gamma n$, for some constant γ. This is maximized when $n_1 = n - 1$ and $n_2 = 0$. This implies that $T_2(n) \leq \alpha n^3 + \beta_2 n^2$. For the case that $v \in f_0$ almost the same analysis works.

2. *A face $f \neq f_0$ with negative charge:* Once we find a face f with negative charge we can find out whether it has a 2-vertex, a tetrad, or it is an M-face, or an MM-face in $O(|f|)$ time.

(a) If f has a 2-vertex the same analysis as in case 1(b) works.

(b) If f has a tetrad we do a BFS which takes $O(n)$ time.

If we find a separating cycle C with size in $\{8, \ldots, 11\}$, we can construct graphs $G_1 = G - \text{In}(C)$ and $G_2 = G - \text{Out}(C)$ in linear time. Assume that n_1 and n_2 are the sizes of G_1 and G_2, respectively. Note that $n_1 + n_2 \leq n + 11$ (because of the size of C) and $9 \leq n_1, n_2 \leq n - 1$ (because C is a separating cycle). Making recursive calls to Procedure 2 on graphs G_1 and G_2 takes $T_2(n_1) + T_2(n_2) \leq \alpha(n_1^3 + n_2^3) + \beta_2(n_1^2 + n_2^2)$ time. This is maximized when one of n_1 or n_2 is equal to $n - 1$ and the other one is 12. Therefore $T_2(n) \leq \alpha[(n-1)^3 + 12^3] + \beta_2[(n-1)^2 + 12^2] + \gamma n$ for some constant $\gamma > 0$. This implies that $T_2(n) \leq \alpha n^3 + \beta_2 n^2$, for large enough α.

If we don't find a separating cycle, then we construct graph G^* which takes at most linear time. Calling Procedure 2 on this graph with size $n-4$ takes $T_2(n-4)$. Then the 3-colouring of G^* can be extended to a 3-colouring of G in constant time by Remark 4.2.7. Therefore, for some constant $\gamma > 0$: $T_2(n) \leq \gamma n + T_2(n-4) \leq \alpha n^3 + \beta_2 n^2$. .

(c) If f is an M-face then we do a BFS which takes linear time. If we find a separating cycle, an analysis almost identical to that of the previous case implies that $T_2(n) \leq \alpha n^3 + \beta_2 n^2$. Otherwise we construct the graph G^* with size $n-6$, which takes linear time. Finding a 3-colouring of G^* takes $T_2(n-6)$ time and extending this colouring to G takes constant time by Remark 4.2.9. Therefore, for some constant $\gamma > 0$: $T_2(n) \leq \gamma n + T_2(n-6) \leq \alpha n^3 + \beta_2 n^2$.

(d) If f is an MM-face, again we spend linear time to do the BFS. If a separating cycle is found as in the analysis of the previous two cases: $T_2(n) \leq \alpha n^3 + \beta_2 n^2$. Otherwise, we construct the graph G^* with size $n-8$ in linear time. Finding a 3-colouring of G^* takes $T_2(n-8)$ time and extending this colouring to G takes constant time by Remark 4.2.11. So for some constant $\gamma > 0$: $T_2(n) \leq \gamma n + T_2(n-8) \leq \alpha n^3 + \beta_2 n^2$, as wanted.

Chapter 5

Colouring the Square of a Planar Graph

Remark 5.0.1 *The results of this chapter are based on papers [41, 42].*

5.1 The Problem and Previous Works

A natural generalization of the 4CP is the following: for a given planar graph G, find the minimum number of colours required in a colouring of the vertices of G such that every two vertices at distance at most two of each other get different colours. This kind of colouring is also referred to in the literature as *distance-2-colouring.* Note that this problem is equivalent to the standard vertex colouring of G^2, the square of graph G.

The question of finding the best possible upper bound for the chromatic number of the square of a planar graph seems to have first been asked by Wegner [58] in 1977. He posed the following conjecture:

Conjecture 5.1.1 *[58] For a planar graph G:*

$$\chi(G^2) \leq \begin{cases} \Delta + 5 & \text{if } 4 \leq \Delta \leq 7, \\ \lfloor \frac{3}{2}\Delta \rfloor + 1 & \text{if } \Delta \geq 8. \end{cases}$$

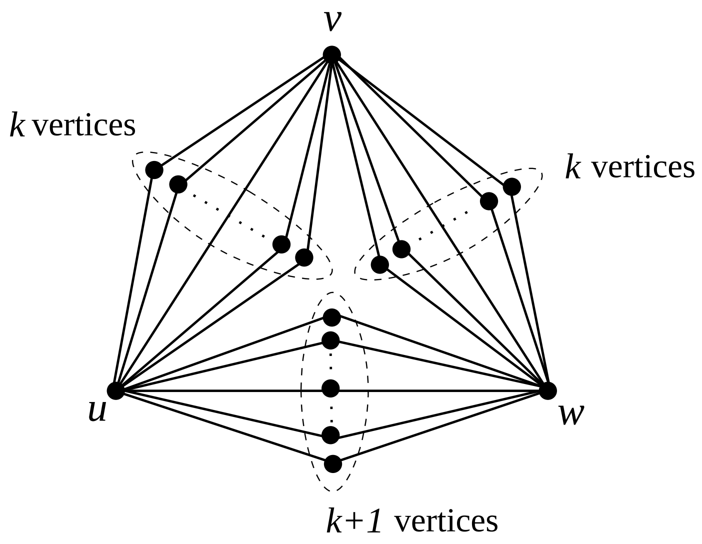


Figure 5.1: A planar graph with $\chi(G^2) = \frac{3}{2}\Delta + 1$

He gave examples illustrating that these bounds are best possible. Figure 5.1 shows such an example for large values of Δ. In this graph, there are k paths of length 2 between u, v and v, w, and $k+1$ paths of length 2 between u, w. So $\Delta = 2k+2$ and all vertices should get different colours. Therefore, $\chi(G^2) = 3k+4 = \lfloor \frac{3}{2}\Delta \rfloor + 1$. He also showed that if $\Delta = 3$ then G^2 can be 8-coloured and conjectured that 7 colours would be enough. Very recently, Thomassen [54] has solved this conjecture for $\Delta = 3$, by showing that the square of every cubic planar graph is 7-colourable, but the conjecture for general planar graphs remains open. This conjecture is mentioned in Jensen and Toft [38], Section 2.18, followed by a brief history of it.

One might think that the straightforward greedy algorithm will give a linear upper bound of approximately 5Δ on $\chi(G^2)$, because every planar graph has a vertex of degree at most 5. But with being more careful in the analysis, one can find out why this argument does not work that easily. For instance, we can argue that since every planar graph G has a vertex with degree at most 5, there is an ordering $v_1, v_2, \ldots, v_n$ of the vertices of G, such that each v_i has at most 5 neighbours in $\{v_1, \ldots, v_{i-1}\}$. This implies a greedy colouring algorithm which uses at most 6 colours to colour G (but not G^2). One may try to extend this argument by saying that, since vertex v_i has at most 5 neighbours in $\{v_1, \ldots, v_{i-1}\}$, it has at most $5(\Delta - 1)$ vertices at distance two in $\{v_1, \ldots, v_{i-1}\}$, and therefore the same algorithm will colour G^2 with at most $5\Delta + 1$ colours. However, the vertices at distance two from v_i in $\{v_1, \ldots, v_{i-1}\}$ are not necessarily adjacent to a vertex in $\{v_1, \ldots, v_{i-1}\}$. So, the number of vertices in $\{v_1, \ldots, v_{i-1}\}$ at distance two from v_i might be much larger than $5(\Delta - 1)$.

Another naive (and failed) argument for showing that $\chi(G^2) \leq 5\Delta+1$ is the following: in any planar graph G there is a vertex v of degree at most 5; by induction there is a colouring C of the square of $G - v$, with at most $5\Delta + 1$ colours. Since there are at most 5Δ vertices at distance at most 2 of v we can assign a colour to v. What's the flaw? Some neighbours of v might have the same colour in C, but they are at distance 2 of each other in G (because of v). So we cannot leave them with their old colours.

The first non-trivial upper bound on $\chi(G^2)$ for each planar graph G was given by Jonas [39] in his Ph.D. thesis, who proved something close to the 5Δ that these failed arguments tried to obtain:

Theorem 5.1.2 *[39] For every planar graph G: $\chi(G^2) \leq 8\Delta - 22$.*

This bound was later improved by Wong in his M.Sc. thesis [60]:

Theorem 5.1.3 *[60] For every planar graph G: $\chi(G^2) \leq 3\Delta + 5$.*

Wong also considered the problem of colouring larger powers of planar graphs and, using the above theorem as the base case of an induction, proved that for every planar graph G and integer $k \geq 1$: $\chi(G^k) \in O(\Delta^{\lfloor \frac{k}{2} \rfloor})$. Note that a rooted tree of height $\lfloor \frac{k}{2} \rfloor$ in which every internal node has degree Δ requires $\Omega(\Delta^{\lfloor \frac{k}{2} \rfloor})$ colours in any colouring of its kth power. Therefore the above bound is asymptotically best possible.

Van den Heuvel and McGuinness [57] gave the following result:

Theorem 5.1.4 *[57] For every planar G: $\chi(G^2) \leq 2\Delta + 25$.*

They also applied the same proof technique to a more generalized setting of colouring, which we will discuss soon. For large values of Δ, Agnarsson and Halldórsson [2] found a better asymptotic bound:

Theorem 5.1.5 *[2] If G is a planar graph with $\Delta \geq 749$, then $\chi(G^2) \leq \lfloor \frac{9}{5}\Delta \rfloor + 2$.*

Recently, Borodin et al. [16, 17] have been able to extend these results further:

Theorem 5.1.6 *[16, 17] For a planar graph G with $\Delta \geq 47$: $\chi(G^2) \leq \lceil \frac{9}{5}\Delta \rceil + 1$.*

In this chapter we give some upper bounds for the chromatic number of the square of a planar graph in terms of the maximum degree, which are asymptotically better than all the previously known bounds. More specifically, we reduce the coefficient of Δ from $\frac{9}{5}$ to $\frac{5}{3}$ and obtain $\chi(G^2)\lceil \frac{5}{3}\Delta \rceil + O(1)$. The main theorem of this chapter is:

Theorem 5.1.7 *For a planar graph G: $\chi(G^2) \leq \lceil \frac{5}{3}\Delta \rceil + 78$.*

For larger values of Δ, we can reduce the additive constant somewhat:

Theorem 5.1.8 *For a planar graph G, if $\Delta \geq 241$, then: $\chi(G^2) \leq \lceil \frac{5}{3}\Delta \rceil + 25$.*

Remark 5.1.9 *The proof of Theorem 5.1.7 is more complicated than the main results of the previous two chapters. That is why we kept this theorem for the last chapter, even though this result was obtained earlier than the previous ones.*

Since the standard vertex colouring for planar graphs [28] and distance-2-colouring for general graphs [32] are both NP-complete, one might expect computing $\chi(G^2)$ for planar G to be NP-complete. Indeed this is true, as proved by Ramanathan and Loyd [46] that the distance-2-colouring problem (and therefore computing $\chi(G^2)$) is NP-complete for planar graphs.

A generalization of standard vertex colouring is $L(p,q)$-labeling. For vertices $u, v \in V$ let $dist(u,v)$ denote the distance between u and v. For integers $p, q \geq 0$, an $L(p,q)$-labeling of a graph G is a mapping $L : V(G) \longrightarrow \{0, \ldots, k\}$ such that

- $|L(u) - L(v)| \geq p$ if $dist(u, v) = 1$, and
- $|L(u) - L(v)| \geq q$ if $dist(u, v) = 2$.

The p, q-span of G, denoted by $\lambda_q^p(G)$, is the minimum k for which an $L(p, q)$-labeling exists. It is easy to see that for any graph G: $\chi(G^2) = \lambda_1^1(G) + 1$. The problem of determining $\lambda_q^p(G)$ has been studied for some specific classes of graphs, such as paths, cycles, wheels, and complete k-partite graphs [32], trees [23, 32], cographs [23], k-almost trees [26], and unicycles and bicycles [39] (See also [9, 27, 29, 30, 31, 46, 45, 56, 59]). The motivation for this problem comes from the channel assignment problem in radio and cellular phone systems, where each vertex of the graph corresponds to a transmitter location, with the label assigned to it determining the frequency channel on which it transmits. In applications, because of possible interference between neighbouring transmitters, the channels assigned to them must have a certain distance from each other. A similar requirement arises from transmitters that are not neighbours but are close, i.e at distance 2. This problem is also known as the Frequency Assignment Problem.

Not surprisingly, computing $\lambda_q^p(G)$ is an NP-hard problem, as the simplest non-trivial case, i.e. $L(1, 0)$-labeling, is the standard vertex colouring of G. The $L(p, q)$-labeling problem, and specially the case $p = 2$ and $q = 1$, has been studied extensively on several classes of graphs (see for example [9, 23, 26, 27, 29, 30, 31, 32]). The $L(2, 1)$-labeling problem is NP-complete for planar, split, chordal, and bipartite graphs [9], and for graphs of diameter 2 [32], and it is polynomially solvable for paths and cycles [32], and trees [23]. However, the complexity of $L(p, q)$-labeling in general is still open for trees.

Because of the motivating application for this problem, it is quite natural to consider it on planar graphs. Since the case $q = 0$ corresponds to labeling the vertices of a graph with integers such that adjacent vertices receive labels at least p apart, the upper bound $3p$ for λ_0^p of planar graphs is easily seen to follow from the Four Colour Theorem. So, let's assume that $q \geq 1$. For any planar graph G, a straightforward argument shows that $\lambda_q^p(G) \geq q\Delta + p - q + 1$. There are planar graphs G (such as the one in Figure 5.1) for which $\lambda_q^p(G) \geq \frac{3}{2}q\Delta + O(p+q)$. The best known upper bound for $\lambda_q^p(G)$, for a planar graph G, is $(4q-2)\Delta + O(p+q)$ proved in [57]:

Theorem 5.1.10 *[57] For any planar graph G and positive integers p and q, such that $p \geq q$:*

$$\lambda_q^p(G) \leq (4q-2)\Delta + 10p + 38q - 24.$$

We sharpen the gap between this result and the best possible bound asymptotically, by obtaining the upper bound $q\lceil\frac{5}{3}\Delta\rceil + O(p+q)$.

Theorem 5.1.11 *For any planar graph G and positive integers p and q:*

$$\lambda_q^p(G) \leq q\lceil\frac{5}{3}\Delta\rceil + 18p + 77q - 18.$$

In [9] Bodlaender et al. give approximation algorithms to compute λ_1^2 for some classes of graphs and noted that the result of Jonas [39] yields an 8-approximation algorithm for planar graphs. Fotakis et al. [27] point out that the result of [57] yields a $(2+o(1))$-approximation algorithm for computing λ_1^1 on planar graphs. Agnarsson and Halldórsson [2] also give a 2-approximation algorithm. Fotakis et al. [27] asks if one can obtain a polynomial time approximation algorithm of approximation ratio < 2. Theorem 5.1.11 answers this question as explained below.

Consider Theorem 5.1.7. It is easy to see that this Theorem yields a $(\frac{5}{3}+\epsilon)$-approximation algorithm for computing $\chi(G^2)$ for any planar graph G, where ϵ is a constant that goes to zero when Δ goes to infinity. Note that this is a trivial approximation algorithm as all we need to do is to compute $\frac{5}{3}\Delta + 78$ and return it. But we actually obtain something more interesting. The proofs of Theorems 5.1.7, 5.1.8, and 5.1.11 are constructive and yield efficient algorithms for finding the corresponding colourings. For example, for Theorem 5.1.7, we obtain an algorithm that produces a distance-2-colouring of any given planar graph G with at most $\frac{5}{3}\Delta + 78$ colours.

The organization of this chapter is as follows. The next section contains the proof of the main Theorem, i.e. Theorem 5.1.7. We start by explaining some of the ideas behind the proof. We formalize these ideas in Subsection 5.2.2 by stating some notation and definitions that will be used throughout the proof, and then describing the reducible configurations. Subsection 5.2.3 explains the set of discharging rules. Finally in Subsection 5.2.4 we complete the proof of the theorem by proving unavoidability of the reducible configurations, using the Discharging Method. In Section 5.3 we show how some simple modifications in the arguments of Section 5.2 yield the proof of Theorem 5.1.8. Then we show in Section 5.4 how to adapt the arguments to prove Theorem 5.1.11. The approximation algorithms obtained based on the proofs of Theorems 5.1.7, 5.1.8, and 5.1.11 are explained in Section 5.5. Finally we talk about the asymptotic tightness of the results of this chapter, if the same set of reducible configurations is used.

5.2 Proof of the Main Theorem

In this section, we give the proof of Theorem 5.1.7 which uses the Discharging Method. Before going into the details of the proof, we explain, very roughly, some of the basic and simple ideas behind this proof and the previously known results.

5.2.1 Going from $\frac{9}{5}\Delta$ to $\frac{5}{3}\Delta$

Let G be an arbitrary planar graph, and assume that G has a very large maximum degree, Δ. Also, assume that we have $\lceil \frac{9}{5}\Delta \rceil + C$ colours to use, for some large constant C (as we said, this is the previously best known upper bound for $\chi(G^2)$ and Borodin et al. [16, 17] proved it for $C = 1$).

The main reducible configuration to prove the bound $\chi(G^2) \leq \lceil \frac{9}{5}\Delta \rceil + C$ is a vertex v with $d_{G^2}(v) \leq \lceil \frac{9}{5}\Delta \rceil + C - 1$, which is adjacent to a vertex u with small degree (say at most 4). Suppose that G has such a vertex v. Then we can *contract* v on edge uv, i.e. remove uv and identify v with u and remove the multiple edges. Call this new graph G'. Since $d(u) \leq 4$, it is easy to see that $\Delta(G') \leq \Delta(G)$, and therefore, we can colour G'

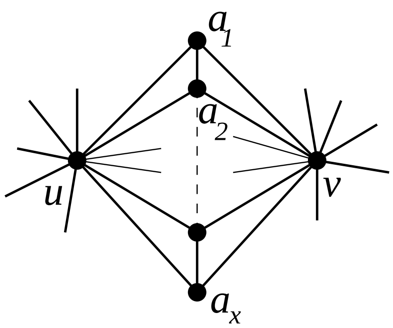


Figure 5.2: Two vertices with many common neighbours

with $\lceil \frac{9}{5}\Delta \rceil + C$ colours. This colouring induced on G can be easily extended to v, since there are at most $\lceil \frac{9}{5}\Delta \rceil + C - 1$ coloured vertices in $N_{G^2}(v)$. We call a vertex like v, a *light* vertex.

So it is enough to show that G has a light vertex. In order to do this, we define two other configurations, each of which contains a light vertex and then prove (using the Discharging Method) that G has at least one of these two configurations.

The first of these two configurations is a ≤ 5-vertex t, all but at most one of whose neighbours have very small degree (say at most 4). In this case, the number of vertices at distance at most two of t is at most $4 \times 4 + \Delta$, which is smaller than $\lceil \frac{9}{5}\Delta \rceil + C$, if Δ is large enough. Therefore t is a light vertex.

For the second configuration, suppose that G is a triangulation. Consider two vertices u and v with large degrees, say Δ, that have x common neighbours. For example, assume that $a_1, \ldots, a_x$ are consecutive (in clockwise order) neighbours of u which are also neighbours of v. Since G is a triangulation, each a_i, $2 \leq i \leq x-1$, has degree exactly 4 and is adjacent to u, v, a_{i-1}, and a_{i+1} (See Figure 5.2). Fix one of these vertices, say a_2, and let's count the number of vertices at distance at most two from it. It is easy to see that $d_{G^2}(a_2) \leq d_G(u) + d_G(v) + d_G(a_1) + d_G(a_3) - x$, since $a_1, \ldots, a_x$ are counted twice, once in $d_G(u)$, and once in $d_G(v)$. Therefore, if $x \geq \frac{\Delta}{5}$, then $d_{G^2}(a_2) \leq \lceil \frac{9}{5}\Delta \rceil$. So a_2 is a light vertex.

If we assume that G is a triangulation then using the Discharging Method one can show that G indeed has a ≤ 5-vertex like t or a 4-vertex like a_2, whose number of neighbours at distance at most two is at most $\lceil \frac{9}{5}\Delta \rceil + C - 1$. Of course, dealing with non-triangulations adds some complications.

As we will see in Section 5.6, there are planar graphs G in which for every vertex v: $d_{G^2}(v) \geq \lceil \frac{9}{5}\Delta \rceil$. Thus, using the idea explained above, we cannot hope for a bound better than $\lceil \frac{9}{5}\Delta \rceil + 1$ and we need to come up with another reducible configuration. This reducible configuration is explained in the next section (Lemma 5.2.14).

5.2.2 Preliminaries and Reducible Configurations

A vertex v is called *big* if $d_G(v) \geq 47$, otherwise we call it a *small* vertex. For this subsection only, we assume that G is a counter-example to Theorem 5.1.7 with the minimum number of vertices. By a colouring we implicitly mean a colouring in which vertices at distance at most two from each other get different colours. Trivially G is connected. The next lemma formalizes the first structure we talked about in the previous subsection.

Lemma 5.2.1 *For every vertex v of G, if there exists a vertex $u \in N(v)$, such that $d_G(v) + d_G(u) \leq \Delta + 2$ then $d_{G^2}(v) \geq \lceil \frac{5}{3}\Delta \rceil + 78$.*

Proof: Assume that v is such a vertex. Contract v on edge uv. The resulting graph has maximum degree at most Δ and because G was a minimum counter-example, the new graph can be coloured with $\lceil \frac{5}{3}\Delta \rceil + 78$ colours. Now consider this colouring induced on G, in which every vertex other than v is coloured. If $d_{G^2}(v) < \lceil \frac{5}{3}\Delta \rceil + 78$ then we can assign a colour to v to extend the colouring to v, which contradicts the definition of G. ■

Recall that by [57]: $\chi(G^2) \leq 2\Delta + 25$. Therefore:

Observation 5.2.2 *We can assume that $\Delta \geq 160$, otherwise $2\Delta + 25 \leq \lceil \frac{5}{3}\Delta \rceil + 78$.*

Lemma 5.2.3 *Every $\leq$5-vertex in G must be adjacent to at least two big vertices.*

Proof: By way of contradiction assume that this is not true. Then there is a $\leq$5-vertex v which is adjacent to at most one big vertex and all its other neighbours are $\leq$46-vertices. Then, using Observation 5.2.2, v along with one of these small vertices will contradict Lemma 5.2.1. ■

Corollary 5.2.4 *Every vertex of G is a $\geq$2-vertex.*

Lemma 5.2.5 *G is 2-connected.*

Proof: By contradiction, let v be a cut-vertex of G and let $C_1, \ldots, C_t$ $(t \geq 2)$ be the connected components of $G - \{v\}$. By the definition of G, for each $1 \leq i \leq t$, there is a colouring φ_i of $G_i = C_i \cup \{v\}$ with $\lceil \frac{5}{3}\Delta \rceil + 78$ colours. We can permute the colours in each φ_i (if needed) such that v has the same colour in all φ_i's and the sets of colours appearing in $N_{G_i}(v)$, $1 \leq i \leq t$, are all disjoint. Now the union of these colourings will be a colouring of G, a contradiction. ■

As mentioned in Subsection 5.2.1, our proof becomes significantly simpler if we can assume that the underlying graph is a triangulation, i.e. all faces are triangles. It will also simplify things to assume that it has minimum degree at least 4. To be able to make these assumptions, we begin by modifying graph G in two phases.

Phase 1: In this phase we transform G into a (simple) triangulated graph G', by adding edges to every non-triangle face of G. Let G' be initially equal to G. Consider any non-triangle face $f = v_1, v_2, \ldots, v_k$ of G'. Because G is 2-connected, we cannot have both $v_1v_3 \in E(G')$ and $v_2v_4 \in E(G')$ at the same time since they both have to be outside of f. So we can add at least one of these edges to $E(G')$ inside f, without creating any multiple edges. We follow this procedure to reduce the faces' sizes as long as we have any non-triangle face in G'. At the end we have a triangulated graph G' which contains G as a subgraph.

Observation 5.2.6 *For every vertex v, $N_G(v) \subseteq N_{G'}(v)$.*

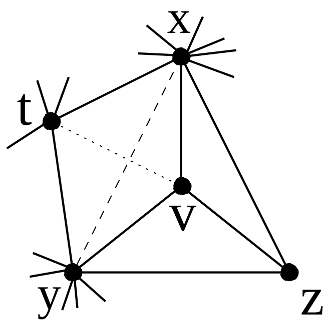


Figure 5.3: The switching operation

Lemma 5.2.7 *All vertices of G' are $\geq$3-vertices.*

Proof: By Corollary 5.2.4 and Observation 5.2.6 all the vertices of G' are $\geq$2-vertices. Suppose that we have a 2-vertex v in G' having neighbours x and y. Since G' is triangulated, the faces on each side of edge vx must be triangles, call them f_1 and f_2. So we must have $xy \in f_1$ and also $xy \in f_2$. Since G' has at least 4 vertices, $f_1 \neq f_2$ and so we have a multiple edge. But G' is simple. ■

Lemma 5.2.8 *Each $\geq$4-vertex v in G' can have at most $\frac{d(v)}{2}$ neighbours which are 3-vertices.*

Proof: Let $x_0, x_1, \ldots, x_{d_{G'}(v)-1}$ be the sequence of neighbours of v in G', in clockwise order. We show that we cannot have two consecutive 3-vertices in this sequence. If there are two consecutive 3-vertices, say $d(x_i) = d(x_{i+1}) = 3$, where addition is in mod $d_{G'}(v)$, then there is a face containing $x_{i-1}, x_i, x_{i+1}, x_{i+2}$. But G' is a triangulated graph. ■

Phase 2: In this phase we transform graph G' into another triangulated graph G'', whose minimum degree is at least 4. Initially G'' is equal to G'. As long as there is any 3-vertex v we do the following *switching* operation: let x, y, z be the three neighbours of v. At least two of them, say x and y, are big in G' by Lemma 5.2.3 and Observation 5.2.6. Remove edge xy. Since G' (and also G'') is triangulated this leaves a face of size 4, say x, v, y, t. Add edge vt to G'' (see Figure 5.3). This way, the graph is still triangulated.

Observation 5.2.9 *If v is not a big vertex in G then $N_G(v) \subseteq N_{G''}(v)$.*

Lemma 5.2.10 *If v is a big vertex in G then $d_{G''}(v) \geq 24$.*

Proof: Follows easily from Lemma 5.2.8 and the definition of the switching operation. ■

So a big vertex v in G will not be a ≤23-vertex in G''. Let v be a big vertex in G and $x_0, x_2, \ldots, x_{d_{G''}(v)-1}$ be the neighbours of v in G'' in clockwise order. We call $x_a, \ldots, x_{a+b}$ (where addition is in mod $d_{G''}(v)$) a *sparse segment* in G'' iff:

- $b \geq 2$,
- Each x_i is a 4-vertex.

In the next two lemmas, let's assume that $x_a, \ldots, x_{a+b}$ is a maximal sparse segment of v in G'', which is not equal to the whole neighbourhood of v. Also assume that x_{a-1} and x_{a+b+1} are the neighbours of v immediately before x_a and immediately after x_{a+b}, respectively.

Lemma 5.2.11 *There is a big vertex in G other than v, that is connected to all the vertices of $x_{a+1}, \ldots, x_{a+b-1}$, in G'' (and in G).*

Proof: Follows easily from Observation 5.2.9, Lemma 5.2.3, and the definition of a sparse segment. ■

We use u to denote the big vertex, other than v, that is connected to all $x_{a+1}, \ldots, x_{a+b-1}$.

Lemma 5.2.12 *All the vertices $x_{a+1}, \ldots, x_{a+b-1}$ are connected to both u and v in G. If x_{a-1} is not big in G then x_a is connected to both u and v in G. Otherwise it is connected to at least one of them. Similarly if x_{a+b+1} is not big in G, x_b is connected to both u and v in G, and otherwise it is connected to at least one of them.*

Proof: Since the only big neighbours of $x_{a+1}, \ldots, x_{a+b-1}$ in G'' are v and u, by Lemma 5.2.3 they must be connected to both of them in G as well. For the same reason x_a and x_{a+b} will be connected to u and v in G, if x_{a-1} and x_{a+b-1} are not big. ■

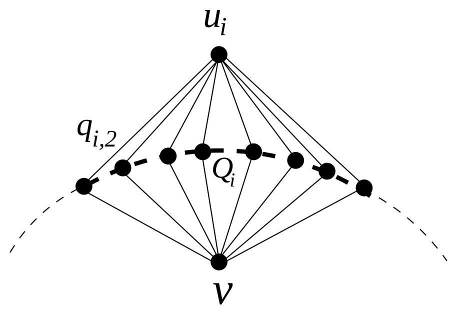


Figure 5.4: The configuration of Lemma 5.2.13

We call $x_{a+1}, \ldots, x_{a+b-1}$ the *inner* vertices of the sparse segment, and x_a and x_{a+b} the *end* vertices of the sparse segment. Consider a vertex v and let us denote the maximal sparse segments of $N(v)$ by $Q_1, Q_2, \ldots, Q_m$ in clockwise order, where $Q_i = q_{i,1}, q_{i,2}, q_{i,3}, \ldots$. The next two lemmas describe the key two reducible configurations for a graph that is a minimum counter-example to the theorem. We have already talked about the first one in Subsection 5.2.1. Here we formalize it.

Lemma 5.2.13 $|Q_i| \leq d_G(v) - \lceil \frac{2}{3}\Delta \rceil - 73$, *for* $1 \leq i \leq m$.

Proof: We prove this by contradiction. Assume that for some i, $|Q_i| > d_G(v) - \lceil \frac{2}{3}\Delta \rceil - 73$. Let u_i be the big vertex that is adjacent to all the inner vertices of Q_i (in both G and G''). See Figure 5.4. For an inner vertex of Q_i, say $q_{i,2}$, we have:

$$\begin{aligned} d_{G^2}(q_{i,2}) &\leq d_G(u_i) + d_G(v) + 2 - (|Q_i| - 3) \\ &\leq \Delta + d_G(v) - |Q_i| + 5 \\ &< \lceil \frac{5}{3}\Delta \rceil + 78. \end{aligned}$$

If $q_{i,2}$ is adjacent to $q_{i,1}$ or $q_{i,3}$ in G then it contradicts Lemma 5.2.1. Otherwise it is only adjacent to v and u_i in G, therefore has degree 2, and so along with v or u_i contradicts Lemma 5.2.1. ■

Lemma 5.2.14 *Consider G and suppose that u_i and u_{i+1} are the big vertices adjacent to all the inner vertices of Q_i and Q_{i+1}, respectively. Furthermore assume that t is a*

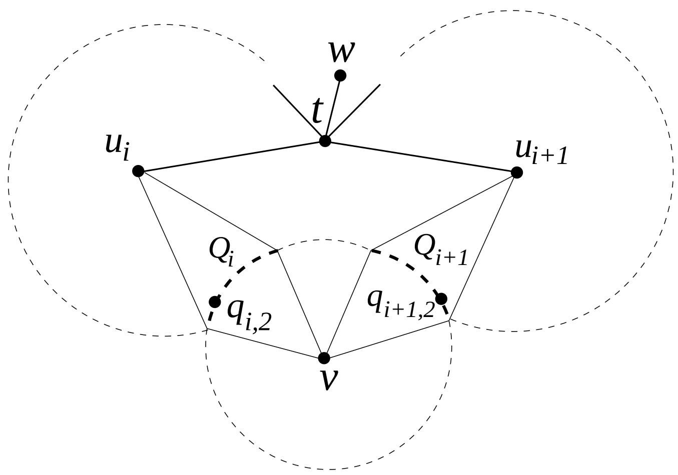


Figure 5.5: Configuration of Lemma 5.2.14

vertex adjacent to both u_i and u_{i+1} but not adjacent to v (see Figure 5.5) and there is a vertex $w \in N_G(t)$ such that $d_G(t) + d_G(w) \leq \Delta + 2$. Let $X(t)$ be the set of vertices at distance at most 2 of t that are not in $N_G[u_i] \cup N_G[u_{i+1}]$. If $|X(t)| \leq 6$ then:

$$|Q_i| + |Q_{i+1}| \leq \lfloor \frac{1}{3}\Delta \rfloor - 67.$$

Proof: Again we use contradiction. Assume that $|Q_i| + |Q_{i+1}| \geq \lfloor \frac{1}{3}\Delta \rfloor - 66$. Using the argument of the proof of Lemma 5.2.1 we can colour every vertex of G other than t. Note that $d_{G^2}(t) \leq d_G(u_i) + d_G(u_{i+1}) + |X(t)| \leq 2\Delta + 6$. If all the colours of the inner vertices of Q_i have appeared on the vertices of $N_G[u_{i+1}] \cup X(t) - Q_{i+1}$ and all the colours of inner vertices of Q_{i+1} have appeared on the vertices of $N_G[u_i] \cup X(t) - Q_i$ then there are at least $|Q_i| - 2 + |Q_{i+1}| - 2$ repeated colours at $N_{G^2}(t)$. So the number of colours at $N_{G^2}(t)$ is at most $2\Delta + 6 - |Q_i| - |Q_{i+1}| + 4 \leq \lceil \frac{5}{3}\Delta \rceil + 76$ and so there is still one colour available for t, which is a contradiction.

Therefore, without loss of generality, there exists an inner vertex of Q_{i+1}, say $q_{i+1,2}$, whose colour is not in $N_G[u_i] \cup X(t) - Q_i$. If there are less than $\lceil \frac{5}{3}\Delta \rceil + 77$ colours at $N_{G^2}(q_{i+1,2})$ then we could assign a new colour to $q_{i+1,2}$ and assign the old colour of it to t and get a colouring for G. So there must be $\lceil \frac{5}{3}\Delta \rceil + 77$ or more different colours at $N_{G^2}(q_{i+1,2})$.

From the definition of a sparse segment $N_G(q_{i+1,2}) \subseteq \{v, u_{i+1}, q_{i+1,1}, q_{i+1,3}\}$. There are at most $d_G(u_{i+1}) + 7$ colours, called the *smaller* colours, at $N_G[u_{i+1}] \cup X(t) \cup N_G[q_{i+1,1}] \cup N_G[q_{i+1,3}] - \{v\} - \{q_{i+1,2}\}$ (note that t is not coloured). So there must be at least $\lceil \frac{2}{3}\Delta \rceil + 70$ different colours, called the *larger* colours, at $N_G[v] - Q_{i+1}$. Since $|N_G[v]| - |Q_i| - |Q_{i+1}| \leq \Delta + 1 - \lfloor \frac{1}{3}\Delta \rfloor + 66 \leq \lceil \frac{2}{3}\Delta \rceil + 67$, one of the *larger* colours must be on an inner vertex of Q_i, which without loss of generality, we can assume is $q_{i,2}$. Because t is not coloured, we must have all the $\lceil \frac{5}{3}\Delta \rceil + 78$ colours at $N_{G^2}(t)$. Otherwise we could assign a colour to t. As there are at most $\Delta + 6$ colours, all from the *smaller* colours, at $N_G[u_{i+1}] \cup X(t)$, all the *larger* colours must be in $N_G[u_i]$, too. Let L be the number of larger colours. Therefore, the number of forbidden colours for $q_{i,2}$ that are not from the larger colours, is at most $d(u_i) - L + d(u_{i+1}) - L \leq 2\Delta - 2L$. By considering the vertices at distance exactly two of $q_{i,2}$ that have a larger colour and noting that $q_{i,2}$ has a larger colour too, the total number of forbidden colours for $q_{i,2}$ is at most $2\Delta - L \leq \lfloor \frac{4}{3}\Delta \rfloor - 70$, and so we can assign a new colour to $q_{i,2}$ and assign the old colour of $q_{i,2}$, which is one of the *larger* colours and is not in $N_{G^2}(t) - \{q_{i+1,2}\}$, to t and extend the colouring to G, a contradiction. ∎

In summary here is the list of reducible configurations we proved in this subsection:

Reducible Configurations:

1. A cut-vertex.

2. A vertex violating Lemma 5.2.1. Such a vertex exists if (but not only if) there exists:

   2(a). a $\leq$5-vertex violating Lemma 5.2.3, or

   2(b). a maximal sparse segment Q_i of a big vertex violating Lemma 5.2.13.

3. Two maximal sparse segments Q_i and Q_{i+1} which contradict Lemma 5.2.14.

In the next two subsections, we prove the unavoidability of this set of configurations. As before, this is done using the Discharging Method. The discharging rules used in this proof are more complicated than the ones we have seen in the previous chapters.

5.2.3 Discharging Rules

Assume that G is an arbitrary planar graph with $\Delta \geq 160$ as the theorem holds for smaller values of Δ by the result of [57]. Our goal is to show that G has at least one of the reducible configurations listed above. If G has reducible configurations 1 or 2(a), we are done. Otherwise, we construct graphs G' and G'' from G as described in the previous subsection. We give an initial charge of $d_{G''}(v) - 6$ units to each vertex v. Using Euler's formula, $|V| - |E| + |F| = 2$, and noting that $3|F(G'')| = 2|E(G'')|$, it is straightforward to check that:

$$\sum_{v \in V} (d_{G''}(v) - 6) = 2|E(G'')| - 6|V| + 4|E(G'')| - 6|F(G'')| = -12. \tag{5.1}$$

By these initial charges, the only vertices that have negative charges are 4- and 5-vertices, which have charges -2 and -1, respectively. The goal is to show that, either G has a reducible configuration listed in the previous subsection or we can send charges from other vertices to $\leq$5-vertices such that all the vertices have non-negative charge, which is of course a contradiction since the total charge must be negative by Equation (5.1).

We call a vertex v *pseudo-big* (in G'') if v is big (in G) and $d_{G''}(v) \geq d_G(v) - 11$. Note that a pseudo-big vertex is also a big vertex, but a big vertex might or might not be a pseudo-big vertex. Before explaining the discharging rules, we more notation.

Suppose that $v, x_1, x_2, \ldots, x_k, u$ is a sequence of vertices such that v is adjacent to x_1, x_i is adjacent to x_{i+1}, $1 \leq i < k$, and x_k is adjacent to u.

Definition: By *"v sends c units of charge through $x_1, \ldots, x_k$ to u"* we mean v sends c units of charge to x_1, it passes the charge to x_2... etc, and finally x_k passes the charge to u. In this case, we also say *"v sends c units of charge through x_1"* and *"u gets c units of charge through x_k"*.

In order to simplify the calculations of the total charges on vertex x_i, $1 \leq i \leq k$, we do not take into account the charges that only pass through x_i. We say v *saves* k units of charge on a set of size h of its neighbours if the net charge loss of v on these neighbours is smaller than h by at least k. More formally, v saves k units of charge on this set if the total charge sent from v to (or through) them minus the total charge sent from (or through) them to v is at most $h-k$ units. For example, if v is sending nothing to u and is getting $\frac{1}{2}$ through u then $h = |\{u\}| = 1$ and the net loss is $0 - \frac{1}{2} = -\frac{1}{2}$. Setting $h-k$ to be equal to the net loss, we get $k = \frac{3}{2}$ and so v saves $\frac{3}{2}$ on u.

In the discharging phase, a big vertex v of G:

1) Sends 1 unit of charge to each 4-vertex u in $N_{G''}(v)$.

2) Sends $\frac{1}{2}$ unit of charge to each 5-vertex u in $N_{G''}(v)$.

In addition, if v is a big vertex and u_0, u_1, u_2, u_3, u_4 are consecutive neighbours of v in clockwise or counter-clockwise order, where $d_{G''}(u_0) = 4$ (see Figure 5.6), then:

3) If $d_{G''}(u_1) = 5$, u_2 is big, $d_{G''}(u_3) = 4$, $d_{G''}(u_4) \geq 5$, and the neighbours of u_1 in clockwise or counter-clockwise order are v, u_0, x_1, x_2, u_2 then v sends $\frac{1}{2}$ to x_1 through u_2, u_1.

4) If $d_{G''}(u_1) = 5$, $5 \leq d_{G''}(u_2) \leq 6$, $d_{G''}(u_3) \geq 7$, and the neighbours of u_1 in clockwise or counter-clockwise order are v, u_0, x_1, x_2, u_2 then v sends $\frac{1}{2}$ to x_1 through u_3, u_2, u_1.

5) If $d_{G''}(u_1) = 5$, u_2 is big, $d_{G''}(u_3) \geq 5$, and the neighbours of u_1 in clockwise or counter-clockwise order are v, u_0, x_1, x_2, u_2 then v sends $\frac{1}{4}$ to x_1 through u_2, u_1.

6) If $d_{G''}(u_1) = 6$, $d_{G''}(u_2) \leq 5$, $d_{G''}(u_3) \geq 7$, and the neighbours of u_1 in clockwise or counter-clockwise order are $v, u_0, x_1, x_2, x_3, u_2$ then v sends $\frac{1}{2}$ to x_1 through u_1.

7) If $d_{G''}(u_1) = 6$, $d_{G''}(u_2) \geq 6$, and the neighbours of u_1 in clockwise or counter-clockwise order are $v, u_0, x_1, x_2, x_3, u_2$ then v sends $\frac{1}{4}$ to x_1 through u_1.

If $7 \leq d_{G''}(v) < 12$ then:

8) If u is a big vertex and $u_0, u_1, u_2, v, u_3, u_4, u_5$ are consecutive neighbours of u where all $u_0, u_1, u_2, u_3, u_4, u_5$ are 4-vertices then v sends $\frac{1}{2}$ to u.

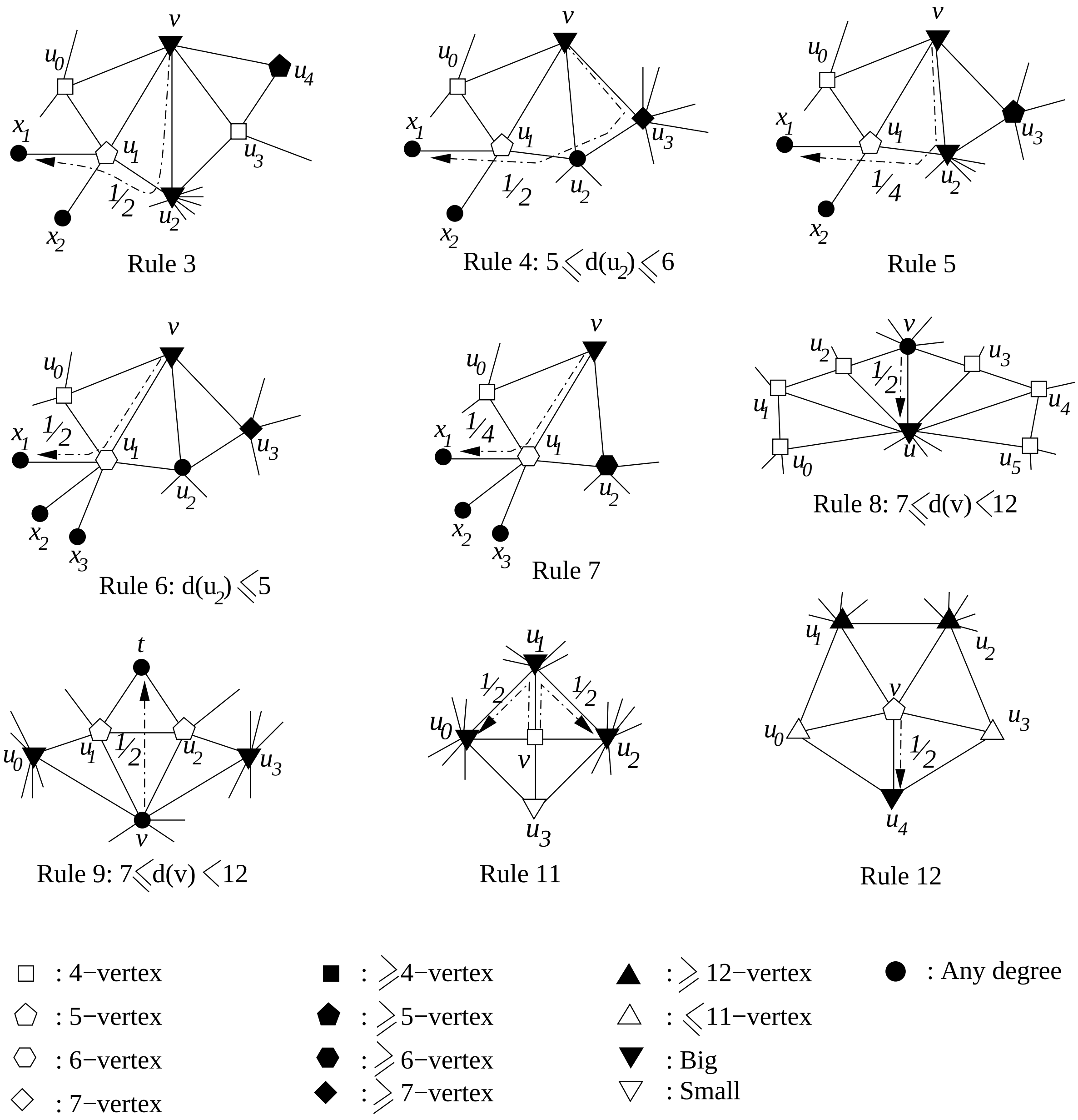


Figure 5.6: Discharging rules

9) If u_0, u_1, u_2, u_3 are consecutive neighbours of v, such that $d_{G''}(u_1) = d_{G''}(u_2) = 5$, u_0 and u_3 are big, and t is the other common neighbour of u_1 and u_2 (other than v), then v sends $\frac{1}{2}$ to t.

Every $\geq$12-vertex v of G'' that was not big in G:

10) Sends $\frac{1}{2}$ to each of its neighbours.

A $\leq$5-vertex v sends charges as follows:

11) If $d_{G''}(v) = 4$ and its neighbours in clockwise order are u_0, u_1, u_2, u_3, such that u_0, u_1, u_2 are big in G and u_3 is small, then v sends $\frac{1}{2}$ to each of u_0 and u_2 through u_1.

12) If $d_{G''}(v) = 5$ and its neighbours in clockwise order are u_0, u_1, u_2, u_3, u_4, such that $d_{G''}(u_0) \leq 11$, $d_{G''}(u_1) \geq 12$, $d_{G''}(u_2) \geq 12$, $d_{G''}(u_3) \leq 11$, and u_4 is big, then v sends $\frac{1}{2}$ to u_4.

From now on, by "the total charge sent from v to one of its neighbours u", we mean the total charge sent from v to u or through u. Similarly, by "the total charge v received from u", we mean the total charge sent from or through u to v.

5.2.4 Details of the Proof

Here we show the unavoidability of the reducible configurations described before. As usual, this is done by establishing a contradiction by calculating the total charge after the discharging phase.

Lemma 5.2.15 *Every big vertex v sends at most $\frac{1}{2}$ to every $5-$ or 6-vertex in $N_{G''}(v)$.*

Proof: For any $5-$ or 6-vertex u, v sends charges to u by at most one rule. ■

Lemma 5.2.16 *If v is big and u_0, u_1, u_2, u_3, u_4 are consecutive neighbours of v in counter-clockwise order, such that $d_{G''}(u_2) \geq 7$ then v sends at most $\frac{1}{2}$ through u_2, or sends 1 through u_2 and $d_{G''}(u_0) = d_{G''}(u_4) = 5$ and u_1 and u_3 are $5-$ or 6-vertices.*

Proof: If u_2 is big and one of rules 3 or 5 applies then it is easy to verify that it is the only rule by which u_2 gets charge from v. If u_1 and u_3 are both 5-vertices then rule 5 may apply twice, one for sending charge to a neighbour of u_1 and one for sending charge to a neighbour of u_3, so overall u_2 gets at most $\frac{1}{2}$ from v. It is straightforward to check that there is no configuration in which we can apply rule 3 twice.

The only other way for v to send charge to u_2 is by rule 4. Note that if this rule applies then none of the other rules apply. Also, v can send charge to u_2 twice by rule 4 since it might apply under clockwise and counter-clockwise orientations of neighbours of v. This happens if $d_{G''}(u_0) = 5$, $5 \leq d_{G''}(u_1) \leq 6$, $5 \leq d_{G''}(u_3) \leq 6$, $d_{G''}(u_4) = 5$, v, u_1, x_2, x_1, x_0 are neighbours of u_0 in clockwise order where $d_{G''}(x_0) = 4$, and y_0, y_1, y_2, u_3, v are neighbours of u_4 in clockwise order where $d_{G''}(y_0) = 4$. In this case v sends $\frac{1}{2}$ to x_1 through u_2, u_1, u_0 and sends $\frac{1}{2}$ to y_1 through u_2, u_3, u_4, and this is the only configuration in which v sends charge to u_2 twice. This proves the lemma. ■

Lemma 5.2.17 *If a vertex v saves a total of at least 6 units of charge on its neighbourhood it will have non-negative charge.*

Proof: If it saves at least 6 units of charge on its neighbourhood, the total net charge sent out from v is at most $d_{G''}(v) - 6$ units of charge, and since the initial charge of v is $d_{G''}(v) - 6$, it will have non-negative charge. ■

Lemma 5.2.18 *Every vertex v that is not big in G will either have non-negative charge after the discharging phase or is reducible configuration 2(a).*

Proof: If v is a 4-vertex it gets a total of at least 2 units of charge by rule 1 and if it is a 5-vertex it gets a total of at least 1 unit of charge by rule 2, unless v is reducible configuration 2(a). Also, the $\leq$5-vertices that send charges by rules 11 and 12 will have non-negative charges, since they are adjacent to at least three $\geq$12-vertices. If $d_{G''}(v) \geq 12$ then it sends $\frac{1}{2}d_{G''}(v) \leq d_{G''}(v) - 6$ by rule 10 and so will have non-negative charge. It is straightforward to verify that there is no configuration in which a 7-vertex

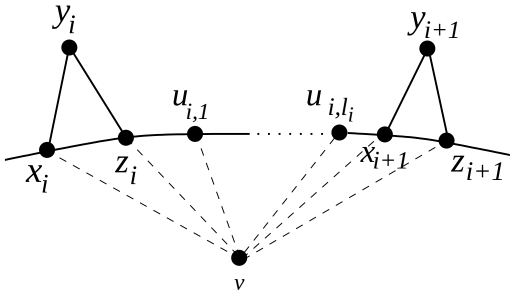


Figure 5.7: Configuration of Lemma 5.2.19

v sends more than 1 unit of charge in rules 8 or 9. Finally, it is not difficult to see that by rules 8 and 9, a vertex sends at most $\frac{1}{2}$ for every two neighbours that it has. So if $8 \leq d_{G''}(v) < 12$ it sends at most $\frac{d_{G''}(v)}{4} \leq d_{G''}(v) - 6$, and therefore it will have non-negative charge in any of these cases. Finally, rules 3 to 7 do not apply to the vertices that are not big in G. ■

Lemma 5.2.19 *Every big vertex v that is not pseudo-big will have non-negative charge.*

Proof: Suppose that v is such a vertex. So $d_{G''}(v) \leq d_G(v) - 12$ and therefore v was involved in at least 12 switching operations, in each of which the edge between v and another big vertex of G was removed. Since G' is simple, these big vertices are distinct. Call them $y_1, y_2, \ldots, y_k$, where $k \geq 12$, in clockwise order. Let $x_i z_i$ be the edge that was added during the switching operation that removed vy_i, and the order of x_i's and z_i's is such that x_i comes before z_i in clockwise order. Note that all x_i's and all z_i's are neighbours of v in G'' (see Figure 5.7).

Let us call the vertices between z_i and x_{i+1}, $u_{i,1}, u_{i,2}, \ldots, u_{i,l_i}$, starting from z_i. For consistency, let us relabel temporarily z_i and x_{i+1} to $u_{i,0}$ and u_{i,l_i+1}, respectively. To show that v saves at least 6 in total, it is enough to show that either v saves at least $\frac{1}{2}$ on a vertex from z_i to x_{i+1}, or v saves at least 1 on the vertices from z_{i+1} to x_{i+2}, for $1 \leq i \leq k$. First we show that there is at least one $\geq$5-vertex in $u_{i,0}, \ldots, u_{i,l_i+1}$, for each $1 \leq i \leq k$. If $u_{i,0}$ is a 4-vertex we must have $y_i u_{i,1} \in G''$, because G'' is a triangulation. Assuming that $u_{i,1}$ is a 4-vertex we must have $y_i u_{i,2} \in G''$ and so on, until we have $y_{i+1} u_{i,l_i+1} \in G''$

and so u_{i,l_i+1} will be a $\geq$5-vertex. So for every $1 \leq i \leq k$, there is a $\geq$5-vertex between z_i and x_{i+1}. Take any such vertex and call it u_{i,j_i}. By Lemmas 5.2.15 and 5.2.16 and rule 10, it can be seen that v saves at least $\frac{1}{2}$ on u_{i,j_i}, unless $7 \leq d_{G''}(u_{i,j_i}) \leq 11$.

So assume that $7 \leq d_{G''}(u_{i,j_i}) \leq 11$ and v sends 1 through u_{i,j_i}. By Lemma 5.2.16 both of the neighbours of v before and after u_{i,j_i} are 5$-$ or 6-vertices and so v saves $\frac{1}{2}$ on them. If $z_i \neq x_{i+1}$ then at least one of these lies between z_i and x_{i+1} and therefore v saves $\frac{1}{2}$ on the vertices from z_i to x_{i+1}. If $z_i = x_{i+1}$ then $u_{i,j_i} = z_i = x_{i+1}$, so $5 \leq d_{G''}(z_{i+1}) \leq 6$ and, $d_{G''}(u_{i+1,1}) = 5$ if $z_{i+1} \neq x_{i+2}$, or $d_{G''}(z_{i+2}) = 5$ otherwise.

First assume that $z_{i+1} = x_{i+2}$. Now if $d_{G''}(z_{i+1}) = 5$ then v gets back $\frac{1}{2}$ from z_{i+1} by rule 12 and so saves 1 on that. If $d_{G''}(z_{i+1}) = 6$ then it is easy to verify that v sends nothing to z_{i+1} by any rule and so saves 1 on that.

Otherwise if $z_{i+1} \neq x_{i+2}$ then there are at least two vertices between $z_{i+1}, \ldots, x_{i+2}$, that are 5$-$ or 6-vertices and so v saves at least $\frac{1}{2}$ on each of them, and therefore saves a total of 1 on the vertices $z_{i+1}, \ldots, x_{i+2}$. ■

So the only vertices that may have negative charges are pseudo-big vertices in G''. Assume that v is a pseudo-big vertex of G'' whose neighbourhood sequence in clockwise order is $x_1, \ldots, x_k$. Let m be the number of maximal sparse segments of the neighbourhood of v and call these segments $Q_1, Q_2, \ldots, Q_m$ in clockwise order. Also, let R_i be the sequence of neighbours of v between the last vertex of Q_i and the first vertex of Q_{i+1}, where $Q_{m+1} = Q_1$. If $m = 0$ then we define R_1 to be equal to $N_{G''}(v)$.

Lemma 5.2.20 *Let $R = x_a, \ldots, x_b$, where R is one of $R_1, \ldots, R_m$. Then v saves at least $\lfloor \frac{|R|}{6} \rfloor$ on the vertices of R.*

Proof: Since R does not overlap with any maximal sparse segment, from every three consecutive vertices x_i, x_{i+1}, x_{i+2} in R (where we consider the neighbours cyclicly if $R = N_{G''}(v)$), at least one of them is a $\geq$5-vertex. Either v sends at most $\frac{1}{2}$ to this vertex, or sends 1 and by Lemma 5.2.16 the two vertices before that and the two vertices after that

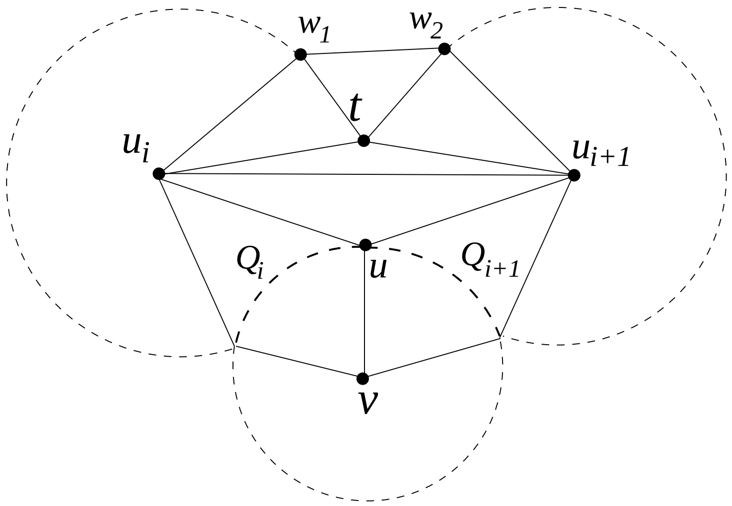


Figure 5.8: The first structure in Lemma 5.2.21

are 5− or 6-vertices and v saves at least $\frac{1}{2}$ on each of them. Thus in either case v saves at least $\frac{1}{2}$ on every three consecutive vertices of R and so saves at least $\lfloor \frac{1}{6}(b-a+1) \rfloor = \lfloor \frac{|R|}{6} \rfloor$.

■

Lemma 5.2.21 *Suppose that $m \geq 4$. Then for every $1 \leq i \leq m$ either v saves at least $\frac{3}{2}$ on R_i, or v saves at least 1 on R_i and*

$$|Q_i| + |Q_{i+1}| \leq \lfloor \frac{1}{3}\Delta \rfloor - 67, \tag{5.2}$$

or G has reducible configuration 2(a) or 3.

Proof: We consider different cases based on $|R_i|$:

$|R_i| = 1$: Assume that $R_i = u$. Since u is the only vertex between two maximal sparse segments, $d_{G''}(u) \geq 5$. First let $d_{G''}(u) = 5$. Since Q_i and Q_{i+1} are sparse segments, there must be two big vertices u_i and u_{i+1} that are connected to all the vertices of Q_i and Q_{i+1}, respectively. Also, u must be connected to these two vertices, because G'' is a triangulation (see Figure 5.8).

Note that by rule 12, v gets back the $\frac{1}{2}$ charge it had sent to u. So v is saving at least 1, so far. Let t be the other vertex that makes a triangle with edge u_iu_{i+1}.

Assume that $d_{G''}(t) = 4$, and w_1, w_2 are the two neighbours of t other than u_i and u_{i+1}. If $d_{G''}(w_1) \leq 4$ and $d_{G''}(w_2) \leq 4$ then since Q_i and Q_{i+1} are sparse segments and u_i and u_{i+1} are big vertices in G, either Equation (5.2) holds, or G has reducible configuration 3. Next assume that $d_{G''}(w_1) \geq 5$. Then by rule 3 u_i will be sending extra $\frac{1}{2}$ to v through u. So overall, v saves $\frac{3}{2}$ on u. If $d_{G''}(t) \geq 5$ then each of u_i and u_{i+1} will send an extra $\frac{1}{4}$ to v through u by rule 5 and therefore v saves $\frac{3}{2}$ on u.

Now let $d_{G''}(u) = 6$, whose neighbours will be v, u_i, u_{i+1}, t, and the end vertices of Q_i and Q_{i+1}. Note that in this case v will send nothing to u and so is saving at least 1. Assume that $d_{G''}(t) = 4$ and its other neighbour is w. If $d_{G''}(w) \leq 6$ then either Equation (5.2) holds, or G has reducible configuration 3. Otherwise, $d_{G''}(w) \geq 7$ and so each of u_i and u_{i+1} sends an extra $\frac{1}{2}$ to v through u by rule 6 and so v saves 2 on u. Next assume $d_{G''}(t) = 5$ and its other neighbours are w_1 and w_2. If $d_{G''}(w_1) \leq 6$ and $d_{G''}(w_2) \leq 6$ then either Equation (5.2) holds, or G has reducible configuration 3. Otherwise at least one of w_1 and w_2 has degree ≥ 7 and so one of u_i or u_{i+1} will send an extra $\frac{1}{2}$ unit of charge to v through u by rule 6 and so v saves $\frac{3}{2}$. If $d_{G''}(t) \geq 6$ then both u_i and u_{i+1} send an extra $\frac{1}{4}$ charge to v through u by rule 7. So v saves $\frac{3}{2}$ on u.

If $7 \leq d_{G''}(u) \leq 11$, or $12 \leq d_{G''}(u)$ and u was not big in G, then u sends $\frac{1}{2}$ to v by rules 8 or 10 and so v saves $\frac{3}{2}$ on u.

If u was big in G then by rule 11 v gets back $\frac{1}{2}$ through u for each of the end vertices of Q_i and Q_{i+1} that are adjacent to u, and so v saves at least 2 on u.

$|R_i| = 2$: Assume that $R_i = v_1, v_2$. If $d_{G''}(v_1) \geq 6$ or $d_{G''}(v_2) \geq 6$ then it is easy to check that v sends nothing to one of v_1, v_2 and sends at most $\frac{1}{2}$ to the other one, or sends $\frac{1}{4}$ to each, and so saves at least $\frac{3}{2}$ on R_i. So let us assume that $d_{G''}(v_1) = d_{G''}(v_2) = 5$ and let t be the other vertex which makes a triangle with

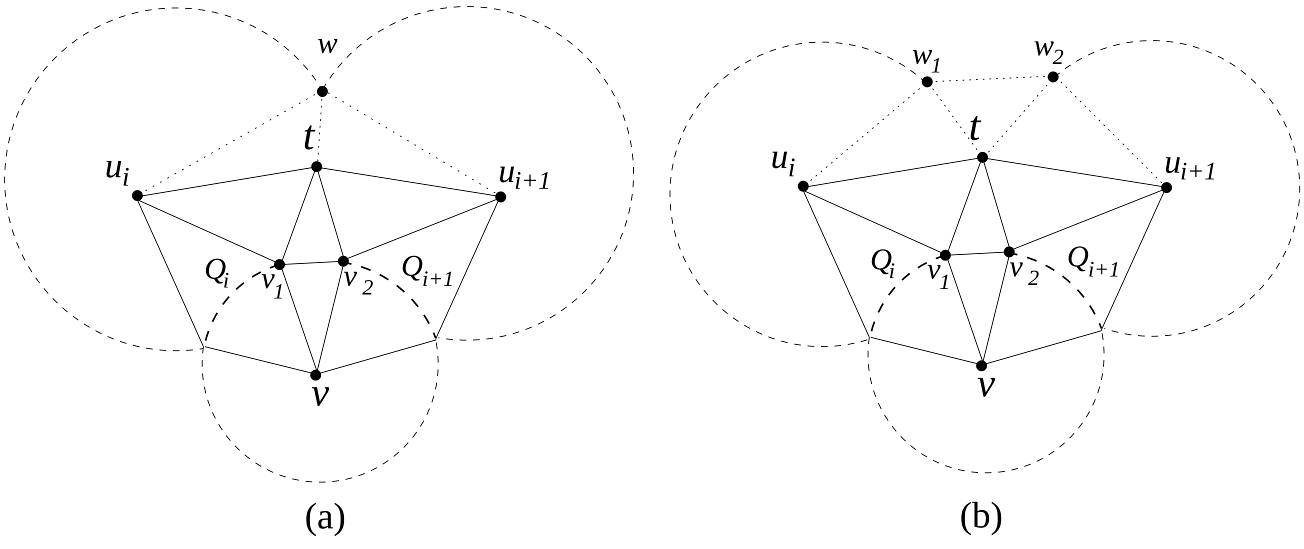


Figure 5.9: Two other structures for Lemma 5.2.21

v_1, v_2. Note that v sends only $\frac{1}{2}$ to each of v_1 and v_2 and so is saving 1 on R_i, so far.

If $d_{G''}(t) = 4$ then either Equation (5.2) holds, or G has reducible configuration 3. Let $d_{G''}(t) = 5$ and call the other neighbour of t (other than u_i, v_1, v_2, u_{i+1}), w (see Figure 5.9(a)). If $d_{G''}(w) \leq 6$ then either Equation (5.2) holds, or G has reducible configuration 3. Otherwise $d_{G''}(w) \geq 7$ and by rule 4 u_i and u_{i+1} each send an extra $\frac{1}{2}$ to v (through v_1 and v_2 respectively) and therefore v saves 2 on R_i. Now let $d_{G''}(t) = 6$ whose neighbours are $w_1, w_2, u_i, u_{i+1}, v_1, v_2$ (see Figure 5.9(b)). If $d_{G''}(w_1) \leq 6$ and $d_{G''}(w_2) \leq 6$ then either Equation (5.2) holds, or G has reducible configuration 3. Otherwise, at least one of w_1 or w_2 is a $\geq$7-vertex and so one of u_i or u_{i+1} sends an extra $\frac{1}{2}$ to v (through v_1 or v_2) by rule 4 and therefore v saves $\frac{3}{2}$ on R_i. If $7 \leq d_{G''}(t) < 12$ then t sends $\frac{1}{2}$ to v by rule 9 and so v saves $\frac{3}{2}$ on R_i. If $12 \leq d_{G''}(t)$ then v gets back the $\frac{1}{2}$ it had sent to each of v_1 and v_2 by rule 12 and so saves at least 2 on R_i.

$|R_i| \geq 3$: If there is no 4-vertex in R_i then they are all $\geq$5-vertices and by Lemmas 5.2.15 and 5.2.16 v saves at least $\frac{3}{2}$ on R_i. If $|R_i| \geq 5$, since R_i cannot have three consecutive 4-vertices, we must have at least three $\geq$5-vertices and again by Lemmas 5.2.15 and 5.2.16 v saves at least $\frac{3}{2}$. So consider the case that $R_i = v_1, v_2, v_3, v_4$, $d_{G''}(v_1) \geq 5$, $d_{G''}(v_4) \geq 5$, and $d_{G''}(v_2) = d_{G''}(v_3) = 4$ (exactly the same argument works for the case that $|R_i| = 3$ and $v_2 = v_3$). There must be a big vertex w, other than v, connected to all the vertices of R_i, or else G has reducible configuration 2(a). If $d_{G''}(v_1) = 5$ then v gets back $\frac{1}{2}$ from v_1 by rule 12 and so saves 1 on v_1. If $d_{G''}(v_1) \geq 6$ it can be verified that v sends nothing to v_1 by any rule and so saves 1 on v_1. Since v saves at least $\frac{1}{2}$ on v_2, it saves at least $\frac{3}{2}$ on R_i.

■

Lemma 5.2.22 *Every pseudo-big vertex v either has non-negative charge or lies in reducible configuration 2(b) or 3.*

Proof: Note that the initial charge of v was $d_{G''}(v) - 6$. So it is enough to show that v saves at least 6 units of charge somewhere in its neighbourhood. We consider different cases based on the value of m, the number of maximal sparse segments of v. Recall that we assume $\Delta \geq 160$.

$m = 0$: Since v is pseudo-big $d_{G''}(v) \geq d_G(v) - 11 \geq 36$. Using Lemma 5.2.20 v will save at least $\lfloor \frac{1}{6} d_{G''}(v) \rfloor \geq 6$ and therefore will have non-negative charge.

$1 \leq m \leq 3$: Either G has reducible configuration 2(b), or Lemma 5.2.13 holds for G. Then by definition of a pseudo-big vertex, if:

- $m = 1$: Then:

$$\begin{aligned} |R_1| &= d_{G''}(v) - Q_1 \\ &\geq d_{G''}(v) - d_G(v) + \lceil \frac{2}{3}\Delta \rceil + 73 \\ &\geq \lceil \frac{2}{3} \times 160 \rceil + 62 \\ &\geq 36. \end{aligned}$$

So by Lemma 5.2.20 v saves at least 6 units of charge on R_1.

- $m = 2$: Then:

$$\begin{aligned}\sum_{1\le i\le 2} |R_i| &= d_{G''}(v) - \sum_{1\le i\le 2} |Q_i| \\ &\ge d_{G''}(v) - 2d_G(v) + 2\times\lceil\frac{2}{3}\Delta\rceil + 146 \\ &\ge \lceil\frac{1}{3}\Delta\rceil + 135 \\ &\ge 36.\end{aligned}$$

So by Lemma 5.2.20 v saves at least 6 units of charge on $R_1 \cup R_2$.

- $m = 3$: Then:

$$\begin{aligned}\sum_{1\le i\le 3} |R_i| &= d_{G''}(v) - \sum_{1\le i\le 3} |Q_i| \\ &\ge d_{G''}(v) - 3d_G(v) + 3\times\lceil\frac{2}{3}\Delta\rceil + 219 \\ &\ge 208.\end{aligned}$$

Therefore by Lemma 5.2.20 v saves at least 6 units of charge on $R_1 \cup R_2 \cup R_3$.

$m = 4$: If v lies in reducible configuration 2(b) or 3 then we are done. So assume that G satisfies Lemmas 5.2.13 and 5.2.14 for v. If v saves $\frac{3}{2}$ on each of $R_1, \dots, R_4$ then it saves 6, and we are done. Otherwise, without loss of generality assume that v saves 1 on R_1 and Equation (5.2) holds for Q_1 and Q_2. Therefore using Lemma 5.2.13:

$$\begin{aligned}|R_2| + |R_3| + |R_4| &\ge d_{G''}(v) - (|Q_1| + |Q_2|) - |Q_3| - |Q_4| \\ &\ge d_{G''}(v) - \lfloor\frac{1}{3}\Delta\rfloor + 67 - 2(d_G(v) - \lceil\frac{2}{3}\Delta\rceil - 73) \\ &\ge \Delta - 2d_G(v) + d_{G''}(v) + 213 \\ &\ge 202.\end{aligned}$$

Thus, by Lemma 5.2.20 v saves at least 6 units on $R_2 \cup R_3 \cup R_4$.

$m = 5$: If G has reducible configuration 2(b) or 3 we are done. Otherwise, G satisfies Lemmas 5.2.13 and 5.2.14 for v. So v saves at least 1 on every R_i, by Lemma 5.2.21. If there are at least two of R_i's such that v saves $\frac{3}{2}$ or more on them then v saves at least 6. Otherwise there is at most one R_i, say R_5, on which v saves at least $\frac{3}{2}$. Therefore Equation (5.2) must hold for $|Q_1| + |Q_2|$ and $|Q_3| + |Q_4|$, i.e:

$$|Q_1| + |Q_2| + |Q_3| + |Q_4| \leq 2 \times \lfloor \frac{1}{3}\Delta \rfloor - 134.$$

Then using Lemma 5.2.13:

$$\begin{aligned} \sum_{1 \leq i \leq 5} |R_i| &\geq d_{G''}(v) - d_G(v) + \lceil \frac{2}{3}\Delta \rceil + 73 - 2 \times \lfloor \frac{1}{3}\Delta \rfloor + 134 \\ &\geq 196. \end{aligned}$$

Therefore v saves at least 6 units of charge on $R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5$, by Lemma 5.2.20.

$m \geq 6$: v saves at least 1 on every R_i, by Lemma 5.2.21. So v saves at least 6 and therefore will have non-negative charge.

■

Proof of Theorem 5.1.7: By Lemmas 5.2.18, 5.2.19, and 5.2.22 every vertex of G'' will either have non-negative charge, after applying the discharging rules, or lie in reducible configuration 2(a), 2(b) or 3. If G has a reducible configuration then we are done. Otherwise the total charge over all the vertices of G'' will be non-negative, but this contradicts Equation (5.1). Therefore G must have one of the reducible configurations listed in Subsection 5.2.2. This disproves the existence of a minimum counter-example to the theorem.

Remark 5.2.23 *Using a more careful analysis one can prove the bound* $\lceil \frac{1}{4}(b - a + 1) \rceil$ *in Lemma 5.2.20, which in turn can be used to prove* $\chi(G^2) \leq \lceil \frac{5}{3}\Delta \rceil + 61$. *By being even more careful throughout the analysis one can probably prove the bound* $\chi(G^2) \leq \lceil \frac{5}{3}\Delta \rceil + 51$ *or even maybe with 30 or 20 instead of 51.*

5.3 A Better Bound for Large Values of Δ

In this section we describe the modifications required to be made to the proof of Theorem 5.1.7 to obtain Theorem 5.1.8. The main steps of the proof of Theorem 5.1.8 are very similar to those of Theorem 5.1.7, and we only have to modify a few lemmas and redo the calculations. For these lemmas, since the proofs are almost identical and do not need any new ideas, we only state the lemmas without giving further proofs. Let G be a minimum counter-example to Theorem 5.1.8 such that $\Delta \geq 241$.

Lemma 5.3.1 *For every vertex v of G, if there exists a vertex $u \in N(v)$, such that $d_G(v) + d_G(u) \leq \Delta + 2$ then $d_{G^2}(v) \geq \lfloor \frac{5}{3}\Delta \rfloor + 25$.*

We construct the triangulated graphs G' and then G'' exactly in the same way. Lemmas 5.2.3 to 5.2.12 are still valid. The analogues of Lemmas 5.2.13 and 5.2.14 will be as follows.

Lemma 5.3.2 $|Q_i| \leq d_G(v) - \lceil \frac{2}{3}\Delta \rceil - 20$*, for* $1 \leq i \leq m$.

Lemma 5.3.3 *Under the same assumption as in Lemma 5.2.14, we have:*

$$|Q_i| + |Q_{i+1}| \leq \lfloor \frac{1}{3}\Delta \rfloor - 14.$$

We apply the same initial charges and discharging rules. Again, all Lemmas 5.2.15 to 5.2.20 hold. The analogue of Lemma 5.2.21 will be:

Lemma 5.3.4 *Suppose that $m \geq 4$. Then for every $1 \leq i \leq m$ either v saves at least $\frac{3}{2}$ on R_i, or v saves at least 1 on R_i and*

$$|Q_i| + |Q_{i+1}| \leq \lfloor \frac{1}{3}\Delta \rfloor - 14,$$

or G has reducible configuration 3.

Now it is straightforward to do the calculations of Lemma 5.2.22 with the above values to see that it holds in this case too. This will complete the proof of Theorem 5.1.8.

5.4 Generalization to Frequency Channel Assignment

In this section we prove Theorem 5.1.11. As we said in Section 5.1, the upper bound $3p$ for λ_0^p of planar graphs follows from the Four Colour Theorem (if we use colours from $\{0, p, 2p, 3p\}$). So let's assume that $q \geq 1$. We prove the following theorem:

Theorem 5.4.1 *For any planar graph G and positive integer p:*

$$\lambda_1^p(G) \leq \lceil \frac{5}{3}\Delta \rceil + 18p + 59.$$

Assuming Theorem 5.4.1, we can prove Theorem 5.1.11 as follows:

Proof of Theorem 5.1.11: Let $c = \lceil \frac{5}{3}\Delta \rceil + 18\lceil \frac{p}{q} \rceil + 60$. By Theorem 5.4.1, there is an $L(\lceil \frac{p}{q} \rceil, 1)$-labeling of G with colours in $\{0, \ldots, c-1\}$. Consider such a labeling and multiply every colour by q. This yields an $L(p, q)$-labeling of G with colours in $\{0, \ldots, q(c-1)\}$. Noting that $q(c-1) \leq q\lceil \frac{5}{3}\Delta \rceil + 18p + 77q - 18$ completes the proof. ■

In the rest of this section we give the proof of Theorem 5.4.1. The steps of the proof are very similar to those of proof of Theorem 5.1.7. Let G be a planar graph which is a counter-example to Theorem 5.4.1 with the minimum number of vertices. We set

$$C = \lceil \frac{5}{3}\Delta \rceil + 18p + 60$$

and throughout this section we use colours from $\{0, \ldots, C-1\}$. Recall that a vertex is said to be big if $d_G(v) \geq 47$.

Lemma 5.4.2 *Suppose that v is a $\leq$5-vertex in G. If there exists a vertex $u \in N(v)$, such that $d_G(v) + d_G(u) \leq \Delta + 2$ then $d_{G^2}(v) \geq d_G(v) + \lceil \frac{5}{3}\Delta \rceil + 73$.*

Proof: Assume that v is such a vertex and assume that $d_{G^2}(v) < d_G(v) + \lceil \frac{5}{3}\Delta \rceil + 73$. Contract v on edge vu. The resulting graph has maximum degree at most Δ and because G was a minimum counter-example, the new graph has an $L(p, 1)$-labeling with at most c colours. Now consider such a labeling induced on G, in which every vertex other than

v is coloured. Every vertex at distance (exactly) two of v in G forbids 1 colour for v, and every vertex in $N(v)$ forbids at most $2p-1$ colours for v. So the total number of forbidden colours for v, i.e. the colours that we cannot assign to v, is at most:

$$\begin{aligned} d_G(v)(2p-1) + d_{G^2}(v) - d_G(v) &< 10p - 5 + \lceil \tfrac{5}{3}\Delta \rceil + 73 \\ &= \lceil \tfrac{5}{3}\Delta \rceil + 10p + 68 \\ &\leq C. \end{aligned}$$

The last inequality follows from the assumption that $p \geq 1$. Therefore, there is still at least one colour available for v whose absolute difference from its neighbours in G^2 is large enough and so we can extend the colouring to G. ■

Observation 5.4.3 *By Theorem 5.1.10 we can assume that $\Delta \geq 162$, otherwise $2(2q-1)\Delta + 10p + 38q - 23 \leq C$.*

Lemma 5.4.4 *Every $\leq$5-vertex must be adjacent to at least 2 big vertices.*

Proof: By way of contradiction assume that there is a $\leq$5-vertex v which is adjacent to at most one big vertex and so all its other neighbours are $\leq$46-vertices. Then, using Observation 5.4.3, v along with one of these small vertices will contradict Lemma 5.4.2.

■

Now construct graph G' from G and then G'' from G' in the same way we did in the proof of Theorem 5.1.7. Also, we define the sparse segments in the same way. Consider vertex v and let's call the maximal sparse segments of it $Q_1, Q_2, \ldots, Q_m$ in clockwise order, where $Q_i = q_{i,1}, q_{i,2}, q_{i,3}, \ldots$.

Lemma 5.4.5 $|Q_i| \leq d_G(v) - \lceil \frac{2}{3}\Delta \rceil - 69$.

Proof: Analogous to the proof of Lemma 5.2.13. ■

The next lemma is analogous to Lemma 5.2.14. The key difference is that we require a bound on the degree of t. This is because each vertex adjacent to t can forbid for t up to $2p-1$ colours. Thus we have to be more careful about controlling the number of such vertices.

Lemma 5.4.6 *Suppose that u_i and u_{i+1} are the big vertices adjacent to all the vertices of Q_i and Q_{i+1}, respectively. Furthermore assume that t is a $\leq$6-vertex adjacent to both u_i and u_{i+1} but not adjacent to v (see Figure 5.5) and there is a vertex $w \in N(t)$ such that $d_G(t) + d_G(w) \leq \Delta + 2$. Let $X(t)$ be the set of vertices at distance at most two of t that are not in $N[u_i] \cup N[u_{i+1}]$. If $|X(t)| \leq 6$ then:*

$$|Q_i| + |Q_{i+1}| \leq \lfloor \frac{1}{3}\Delta \rfloor - 60. \tag{5.3}$$

Proof: Again, by way of contradiction, assume that $|Q_i| + |Q_{i+1}| \geq \lfloor \frac{1}{3}\Delta \rfloor - 59$. Using the same argument as at the beginning of the proof of Lemma 5.4.2, we can colour every vertex of G other than t using colours in $\{0, \dots, C-1\}$ such that the vertices that are adjacent receive colours that are at least p apart and the vertices at distance two receive distinct colours. Consider such a colouring.

Remark: We often focus on the inner vertices of Q_i. So recall that there are exactly $|Q_i| - 2$ such vertices (similarly for Q_{i+1}).

Claim 1: There are at least $\lceil \frac{5}{3}\Delta \rceil + 78$ colours in $N_{G^2}(t)$ and they forbid all the C colours for t.

Proof: Trivially, if there is a non-forbidden colour for t then we can extend the colouring to t, which contradicts the minimality of G.

If there are at most $\lceil \frac{5}{3}\Delta \rceil + 77$ colours in $N_{G^2}(t)$ then (because t is not coloured and has degree at most 6) they forbid at most $\lceil \frac{5}{3}\Delta \rceil + 71 + 6(2p-1) = \lceil \frac{5}{3}\Delta \rceil + 12p + 65 < C$ colours for t, which contradicts what we proved in the previous paragraph. ∎

Claim 2: There exists an inner vertex of Q_i or Q_{i+1} whose colour is distinct from the colour of every other vertex in $N_{G^2}(t)$ and differs from the colour of every vertex in $N(t)$ by at least p.

Proof: By way of contradiction assume the above statement is false. Let us count the number of forbidden colours for t. The neighbours of t forbid at most $d_G(t) \times (2p-1)$ colours for t. Let's denote this set of forbidden colours by R. The vertices at distance exactly two of t are in $N(u_i) \cup N(u_{i+1}) \cup X(t) - N(t)$, and each of them forbids its own colour for t. However, at least $|Q_i| - 2 + |Q_{i+1}| - 2$ of these forbidden colours (for t) are counted twice. This is because we assumed the claim is false; i.e. for every colour α that appears on an inner vertex of Q_i or Q_{i+1} there is a neighbour of t whose colour differs from α by less than p (and so $\alpha \in R$) or there is another vertex in $N_{G^2}(t)$ with colour α. Since $d_{G^2}(t) \leq d_G(u_i) + d_G(u_{i+1}) + |X(t)| \leq 2\Delta + 6$, the total number of forbidden colours for t is at most $d_G(t) \times (2p-1) + 2\Delta + 6 - d_G(t) - |Q_i| - |Q_{i+1}| + 4 \leq \lceil\frac{5}{3}\Delta\rceil + 6(2p-1) + 63 \leq \lceil\frac{5}{3}\Delta\rceil + 12p + 57 < C$. This contradicts Claim 1. ■

Without loss of generality, assume there exists an inner vertex of Q_{i+1}, say $q_{i+1,2}$, whose colour is different from the colour of every vertex in $N_{G^2}(t)$ and differs from the colour of every vertex in $N(t)$ by at least p.

Claim 3: There are at least $\lceil\frac{5}{3}\Delta\rceil + 77$ colours in $N_{G^2}(q_{i+1,2})$ and they forbid for $q_{i+1,2}$, $C-1$ colours (all the colours except the one that appears on $q_{i+1,2}$).

Proof: First we show that the vertices in $N_{G^2}(q_{i+1,2})$ must forbid all the colours (except the one that appears on $q_{i+1,2}$) for $q_{i+1,2}$. Otherwise, we can remove the colour of $q_{i+1,2}$ and assign it without any conflict to t (because Claim 2 holds), and assign a new colour (from the colours that are not forbidden) to $q_{i+1,2}$. Hence, the number of forbidden colors for $q_{i+1,2}$ is $C-1$.

If there are fewer than $\lceil\frac{5}{3}\Delta\rceil+77$ different colours in $N_{G^2}(q_{i+1,2})$ then, since $d_G(q_{i+1,2}) \leq 4$, the vertices in $N_{G^2}(q_{i+1,2})$ forbid fewer than $4(2p-1) + \lceil\frac{5}{3}\Delta\rceil + 73 = \lceil\frac{5}{3}\Delta\rceil + 8p + 69 \leq C-1$ colours for $q_{i+1,2}$. This contradicts what we proved in the previous paragraph. ■

From the definition of a sparse segment $N(q_{i+1,2}) \subseteq \{v, u_{i+1}, q_{i+1,1}, q_{i+1,3}\}$. Let's denote the set of colours on the vertices in $N[u_{i+1}] \cup N(t) \cup X(t) \cup N[q_{i+1,1}] \cup N[q_{i+1,3}]$ by S and call it the set of *smaller colours*.

Claim 4: $|S| \leq d_G(u_{i+1}) + 14$.

Proof: Follows from the definition of S. ■

Let us call the set of colours that are forbidden for t or $q_{i+1,2}$ by the smaller colours the *smaller forbidden* colours, and denote them by SF. Since $d(t) \leq 6$ and $d(q_{i+1,2}) \leq 4$ and u_{i+1} is a common neighbour of t and $q_{i+1,2}$,

$$|SF| \leq 9(2p-1) + |S| - 9 = |S| + 18p - 18. \tag{5.4}$$

So, SF contains S along with at most $18(p-1)$ colours which differ from the colour of some neighbour of t or some neighbour of $q_{i+1,2}$ by at most $p-1$.

Claim 5: Every colour that is not in SF differs from every colour in $N(t) \cup N(q_{i+1,2})$ by at least p.

Proof: By the definition of SF, every colour which differs from the colour of a vertex in $N(t) \cup N(q_{i+1,2})$ by less than p is in SF. ■

We will use Claim 5 at the end of the proof of this Lemma. By Claim 3, there are at least $C - 1 - |SF|$ colours, different from the smaller forbidden colours, in $N(v) - Q_{i+1}$. We call this set the *larger* colours and denote it by L.

Claim 6: $|L| \geq \lceil \frac{5}{3}\Delta \rceil - |S| + 77 \geq \lceil \frac{5}{3}\Delta \rceil - d_G(u_{i+1}) + 63$.

Proof: Follows from the definition of L, Claim 4, and the bound on $|SF|$ (Inequality 5.4). ■

Since $|N(v)| - (|Q_i| - 2) - |Q_{i+1}| \leq \Delta - \lfloor \frac{1}{3}\Delta \rfloor + 61 < |L|$, one of the *larger* colours must be on an inner vertex of Q_i, which without loss of generality, we can assume is $q_{i,2}$.

Claim 7: The vertices in $N(v) - Q_{i+1} - \{q_{i,2}\}$ forbid for $q_{i,2}$ all the colours in L, except the one that appears on $q_{i,2}$.

Proof: All the larger colors appear in $N(v) - Q_{i+1}$ and so they are at distance at most two of $q_{i,2}$. ■

Claim 8: The number of forbidden colours for $q_{i,2}$ is at most $\lfloor \frac{4}{3}\Delta \rfloor + 8p - 68 < C$.

Proof: By noting that $d(q_{i,2}) \leq 4$, neighbours of $q_{i,2}$ forbid at most $4(2p-1)$ colours for $q_{i,2}$. Now let's count the number of forbidden colours for $q_{i,2}$ by the vertices at distance exactly two of it.

Since the colours in $N[u_{i+1}] \cup N(t) \cup X(t)$ are smaller colours and forbid for t only colours that are in SF, by Claim 1, all the larger colours must appear in $N[u_i] - N(t)$. Remember that the larger colours appear in $N(v) - Q_{i+1}$, too. Therefore, the number of colours that are not in L and are forbidden for $q_{i,2}$ by the vertices at distance exactly 2 of $q_{i,2}$ is at most: $d(u_i) - 1 - (|L| - 1) + d(v) - 1 - (|L| - 1) \leq 2\Delta - 2|L|$. By considering the vertices at distance exactly two of $q_{i,2}$ that have a larger colour and noting that $q_{i,2}$ has a larger colour too, and using Claim 6, the total number of forbidden colours for $q_{i,2}$ is at most:

$$\begin{aligned} 4(2p-1) + (2\Delta - 2|L|) + (|L| - 1) &\leq \lfloor \frac{1}{3}\Delta \rfloor + d_G(u_{i+1}) + 8p - 68 \\ &\leq \lfloor \frac{4}{3}\Delta \rfloor + 8p - 68. \end{aligned}$$

■

By Claim 8, we can assign the colour of $q_{i,2}$ to t (because it is a larger colour and so it is different from the colours in $X(t)$ and, by Claim 5, differs from all the colours in $N(t)$ by at least p) and find a new colour for $q_{i,2}$ that is not forbidden for it. ■

The rest of the proof is almost identical to that of Theorem 5.1.7. We use Lemmas 5.4.4, 5.4.5, and 5.4.6, instead of Lemmas 5.2.3, 5.2.13, and 5.2.14, respectively. The initial charges and the discharging rules are the same. Without any modifications, Lemmas 5.2.15 to 5.2.20 hold in this case, too. In Lemma 5.2.21 we should replace Equation (5.2) with Equation (5.3) and use Lemma 5.4.6 instead of Lemma 5.2.14. To do so, it is important to note that whenever we used Lemma 5.2.14 in the proof of Lemma 5.2.21, the degree of t was at most 6; thus, we can use Lemma 5.4.6, instead. After doing these modifications, the calculations for the proof of this revised version of Lemma 5.2.21 are fairly straightforward.

5.5 The Colouring Algorithms

In this section we show how to transform the proof of Theorem 5.1.7 into an algorithm. that colours the vertices of a given embedded planar graph G with $\frac{5}{3}\Delta + 78$ colours such that every pair of vertices at distance at most two from each other get different colours. Since in any proper colouring of G^2 we need at least $\Delta+1$ colours this will be a $(\frac{5}{3}+o(1))$-approximation algorithm, for large enough values of Δ. With some minor modifications in the algorithm, we can obtain colouring algorithms for Theorems 5.1.8 and 5.1.11.

Consider a planar graph G. We may assume that $\Delta \geq 160$ since for smaller values of Δ it is straightforward to obtain an algorithm based on the result of [57] that uses at most $\lceil \frac{5}{3}\Delta \rceil + 78$ colours. Also, we assume that the input to our algorithm is connected, since for a disconnected graph it is enough to colour each connected component, separately. One iteration of the algorithm either finds a cut-vertex and breaks the graph into smaller subgraphs, or reduces the size of the problem by contracting a suitable edge of G. Then it colours the new smaller graph(s) recursively, and then extends the colouring(s) to G. More specifically, we do the following steps, as long as the graph has at least one vertex:

1. Check to see whether G has a cut-vertex. If v is a cut-vertex and $C_1, \ldots, C_k$ are connected components of $G - v$ then colour each $G_i = C_i \cup \{v\}$, independently. The union of these colourings, after permuting the colours in some of them will be a colouring of G.

2. Else, check to see whether there is a $\leq$5-vertex adjacent to at most one big vertex. If such a vertex exists, then that vertex along with one of its small neighbours will be the suitable edge to be contracted.

3. Else, construct the triangulated graph G''.

4. Apply the initial charges and the discharging rules.

5. As the total charge is negative, we can find a vertex v with negative charge. This vertex must have or lie in one of reducible configurations 2(a), 2(b) or 3.

   If v is reducible configuration 2(a) then we continue as explained in the second step. If we find reducible configuration 2(b) around v then one of the inner vertices of the sparse segment along with one of its two big neighbours will be the suitable edge to contract. Finally, if we find reducible configuration 3 around v then we can contract edge tw (recall the specification of t and w from Lemma 5.2.14).

6. Colour the new graph (after contracting the suitable edge), recursively.

7. This colouring can be easily extended to G by the arguments of proofs of Lemmas 5.2.3, 5.2.5, 5.2.13 or 5.2.14.

For a given graph G let $n = |V|$ be the size of G, and denote the worst case running time of the algorithm for an input of size n by $T(n)$. We prove by induction that for all values of n and for some constant $C > 0$: $T(n) \leq Cn^2$. The inequality is trivial for small values of n. So let's assume that $T(i) \leq Ci^2$ for $1 \leq i < n$ and consider the case that the input graph has size n.

Finding a cut-vertex in a graph takes linear time. Once we have done that we make recursive calls on k smaller graphs $G_1, \ldots, G_k$, with $2 \leq k \leq n-1$. Let $n_i = |V(G_i)|$, $1 \leq i \leq k$. Note that $2 \leq n_i \leq n-1$ (for $1 \leq i \leq k$) and $\sum_{i=1}^{k}(n_i - 1) = n - 1$. Therefore, for some constant $\alpha > 0$: $T(n) \leq \alpha n + \sum_{i=1}^{k} T(n_i) \leq \alpha n + C\sum_{i=1}^{k} n_i^2$. The last summation is maximized when $k = 2$ and one of n_1 or n_2 is equal to $n - 1$. This easily implies that $T(n) \leq Cn^2$.

To do the second step we go through all ≤ 5-vertices and check the degree of their neighbours. This can be easily done in $O(n)$.

To construct graph G' we spend at most $O(|f|)$ time on every face f. So it takes $O(\sum_{f\in F} |f|)$ time to make G', which is in $O(n)$. To construct G'' we should do at most $O(n)$ switching operations, each of which takes constant time.

Applying the initial charges can be done in linear time, too. For each vertex v, it takes at most $O(d_{G''}(v))$ to apply the discharging rules to it. So, applying the discharging rules takes $O(\sum_{v \in V} d_{G''}(v))$ time, which is linear in n. Finding a vertex v with negative charge can be done in $O(n)$ time. Finding a suitable edge to contract around v takes at most $O(n)$ time.

Once the suitable edge is found (in step 2 or 5) it takes at most $O(n)$ time to contract it. After finding the colouring of the new graph, it takes at most $O(n)$ time to extend this colouring to G using the arguments of the proofs of Lemmas 5.2.3, 5.2.13 or 5.2.14. Therefore, for some constant $\alpha > 0$: $T(n) \leq \alpha n + T(n-1) \leq \alpha n + C(n-1)^2 \leq Cn^2$, as wanted.

The algorithms for Theorems 5.1.8 and 5.1.11 work almost identically.

5.6 On Possible Asymptotic Improvements of the Main Theorem

In this section, we only focus on the asymptotic order of the bounds, i.e. the coefficient of Δ. As we said in Subsection 5.2.1, the main reducible configuration to prove the bound $\chi(G^2) \leq \frac{9}{5}\Delta + O(1)$ for planar G, is a vertex v with at most $\frac{9}{5}\Delta + O(1)$ vertices in $N_{G^2}(v)$. The results of [2] and [16, 14] are essentially based on showing that every planar graph has such a vertex. However, as pointed out in [2] and [16, 14], this is the best possible bound on the minimum degree of G^2. That is, there are 2-connected planar graphs in which every vertex v satisfies $d_{G^2}(v) \geq \lceil \frac{9}{5}\Delta \rceil$. One of these extremal graphs can be obtained from the icosahedron, by taking a perfect matching, adding $k-1$ paths of length two parallel to each edge of the perfect matching, and replacing every other edge of the icosahedron by k parallel paths of length two (see Figure 5.10).

Therefore, by only bounding the minimum degree of G^2 we cannot improve the bound $\frac{9}{5}\Delta + O(1)$, asymptotically. This is the reason we introduced reducible configuration 3.

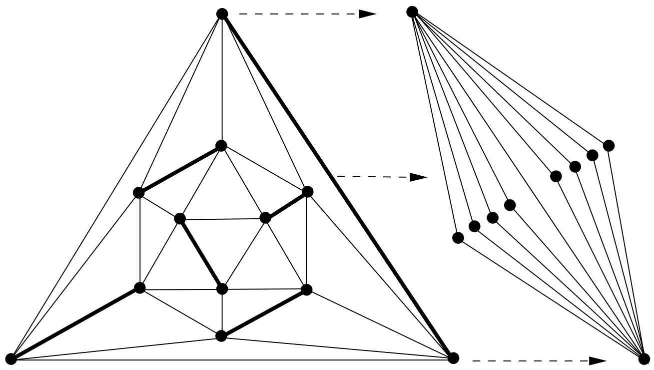


Figure 5.10: The icosahedron and the modified graph

We proved that any planar graph G either has a cut-vertex, or a vertex v such that $d_{G^2}(v) \leq \frac{5}{3}\Delta + O(1)$, or has configuration 3.

But there are graphs that are extremal for this new set of reducible configurations in the following sense: these graphs do not have a cut-vertex, do not have a vertex v with $d_{G^2}(v) \leq \frac{5}{3}\Delta$, and do not have configuration 3. For an odd value of k, one of these graphs, which is obtained from a tetrahedron, is shown in Figure 5.11. To interpret this figure, we have to join the three copies of v_8 and remove the multiple edges (we draw the graph in this way for clarity). Also, the dashed lines represent sequences of consecutive 4-vertices. Around each of $v_1, \ldots, v_4$ there are $3k - 6$ such vertices. So, $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 3k$, $d(v_5) = d(v_6) = d(v_7) = d(v_8) = 3k + 3$, $\Delta = 3k + 3$, and for any vertex $v \in G$: $d_{G^2}(v) \geq 5k + 3$ (with equality holding for $v \in \{v_1, \ldots, v_4\}$). The minimum degree of G^2 is $\frac{5}{3}\Delta + O(1)$ and it is easy to see that G does not have configuration 3. Therefore, using reducible configurations similar to those of Subsection 5.2.2 the best asymptotic bound that we can achieve is $\frac{5}{3}\Delta + O(1)$. So we need another reducible configuration to improve the multiplicative constant $\frac{5}{3}$.

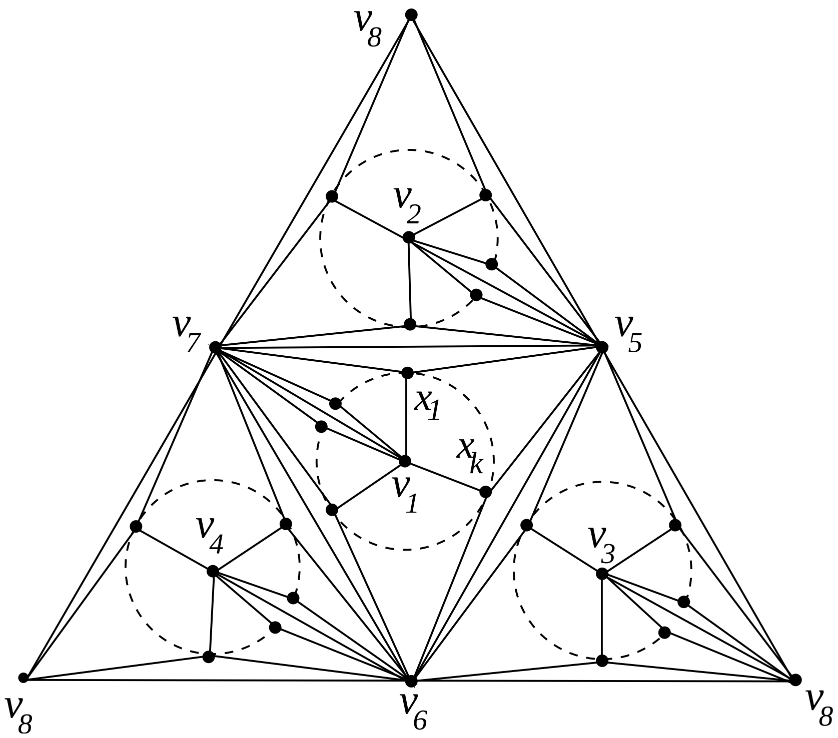


Figure 5.11: The extremal graph for reducible configurations 2 and 3

Chapter 6

Concluding Remarks

In this thesis we studied two colouring problems on planar graphs and used the Discharging Method to improve the previously best known result on each of them. The first problem is Steinberg's conjecture, which states that every planar graph without cycles of size 4 and 5 is 3-colourable. We proved that planar graphs without cycles of size in $\{4, \ldots, 7\}$ are three colourable. The second problem is a conjecture by Wegner, which states that the square of any planar graph G can be coloured with at most $\lfloor \frac{3}{2}\Delta \rfloor + 1$ colours. We improved the previously best known bound on the chromatic number of the square of a planar graph G by showing that $\chi(G^2) \leq \lceil \frac{5}{3}\Delta \rceil + O(1)$.

However, both of these conjectures (by Steinberg and Wegner) remain open. In this chapter, along with these two major conjectures, we talk about several open problems. Some of these problems are on possible improvements on the results we have obtained in this thesis, with the hope of proving these two conjectures. These problems are the more difficult problems we present. We also discuss some open problems related to these two conjectures whose study might shed some light on paths toward resolving these conjectures. Some of these problems seem to be easier than the former ones and have not been studied seriously either in the literature or by the author.

6.1 On 3-Colouring Planar Graphs and Steinberg's Conjecture

The next step toward the conjecture of Steinberg is to prove that planar graphs without cycles of size in $\{4, 5, 6\}$ are 3-colourable. We believe that by combining the ideas of Chapters 3 and 4 and considering some more complicated reducible configurations (similar to those in Chapter 3), which involve interactions of two or more faces, we might be able to do this step. The main difficulty in this line of attack would be, of course, in dealing with faces of size 7. Therefore, most of the new reducible configurations would probably involve 7-faces. To prove Steinberg's full conjecture using this approach we would probably have to consider many more reducible configurations, so many so that a computer-aided proof seems unavoidable.

Another, perhaps easier, step to consider is Steinberg's conjecture under the extra condition that every two triangles in the graph are far from each other. More specifically, for a planar graph G, let $d(G)$ denote the minimum distance between two triangles in G, given by the number of edges in a shortest path joining two triangles in G. If $d(G) > 0$ (say at least 1 or 2) and G does not have 4- and 5-cycles, is it true that G is 3-colourable? This weaker version of Steinberg's conjecture seems easier to prove since many of the reducible configurations we may need to consider to prove Steinberg's full conjecture involve adjacent triangles (two triangles sharing a vertex) or triangles that are close to each other. For instance, if we assume $d(G)$ is large enough, then we can bound from above the number of bad vertices (3-vertices incident with a triangle) incident with ≤ 12-faces. This will be quite helpful in the discharging phase (recall the proofs of Example 2.2.4 and Theorem 3.2.1). Therefore, if we put a lower bound on the distance between triangles, that will bring down the number of reducible configurations significantly.

The problem suggested above is also a weaker version of an open problem discussed in Jensen and Toft [38] (Problem 2.10): If G is a planar graph with finite, but sufficiently large (say 4 or 5) $d(G)$, is G then 3-colourable? Note that here we do not have the restriction of not having 4- and 5-cycles. If $d(G) = \infty$ then there is at most one triangle in each component of G and by the theorem of Grötsch [33] and an extension by Grünbaum [34] and by Aksinov [3], G is indeed 3-colourable. It is known that $d(G) \leq 3$ is not sufficient, as there are planar graphs with $d(G) = 3$ that are not 3-colourable.

6.2 On Distance-2-Colouring and Related Problems

As we mentioned in Remark 5.2.23, the additive constant in the bound $\chi(G^2) \leq \lceil \frac{5}{3}\Delta \rceil + O(1)$ can be reduced somewhat by doing a more careful analysis of the total charges after the discharging phase. But it is not clear how to bring this constant down close to 1 (say below 10). However, reducing the additive constant does not seem as interesting nor as important as improving this bound asymptotically.

As discussed in Section 5.6, to improve this bound (and the other two theorems of Chapter 5) asymptotically and possibly prove Wegner's conjecture (using the Discharging Method), we have to find a new reducible configuration, different from configurations 1-3 listed in Subsection 5.2.2. We do not know exactly what the structure of a new reducible configuration should look like, but one thing that we know is that this new configuration must exist in the graph of Figure 5.11. The reason is that this graph is 2-connected (so does not have configuration 1) and neither has a vertex v with $d_{G^2}(v) \leq \frac{5}{3}\Delta$ nor has configuration 3. Therefore, the best way to find a new reducible configuration is to look at the extremal graph of Figure 5.11, since if it exists at all this graph must have it.

There seems to be a close relation between distance-2-colouring and another type of colouring, called *cyclic colouring* (discussed below). As we explain soon, studying the cyclic colouring problem might help to find a new reducible configuration for the distance-2-colouring problem and improve the results of Chapter 5, asymptotically.

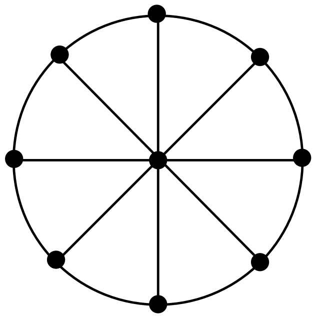


Figure 6.1: A wheel graph

6.2.1 Cyclic Colourings of Planar Graphs

Consider an embedded planar graph $G(V, E)$ with face set F. Define a new set of vertices V^* by putting a vertex v_f in V^* for every face $f \in F$. Also, create a new edge set E^* as follows: for every edge $uv \in E$ consider the two (not necessarily distinct) faces f and f' that are on the two sides of uv. Let $v_f, v_{f'} \in V^*$ be the vertices corresponding to these faces. Put the edge $v_f v_{f'}$ into E^*. The new graph $G^*(V^*, E^*)$ is called the *dual* graph of G. Note that G^* is not necessarily simple as it may have loops (if G has bridges) or multiple edges (if two faces in G share more than one edge). It is easy to see that the dual graph is also a planar graph, and the dual graph of G^* is G.

The 4CP was originally stated as follows: the number of colours required to colour the *faces* of an arbitrary planar graph in such a way that, two distinct faces which are incident with the same edge receive different colours, is at most 4. Note that this is equivalent to colouring the vertices of the dual graph.

In 1969, Ore and Plummer [43] defined a new type of face colouring of planar graphs, more restrictive than the one in 4CP. A face colouring is *angular* if two distinct faces which are incident with the same *vertex* receive distinct colours. Equivalently, we want to colour a map of countries, such that two countries that share even a point on their borders (and not necessarily a line segment) receive different colours. The *angular* chromatic number of G is the minimum number of colours required in any angular colouring of G. Clearly,

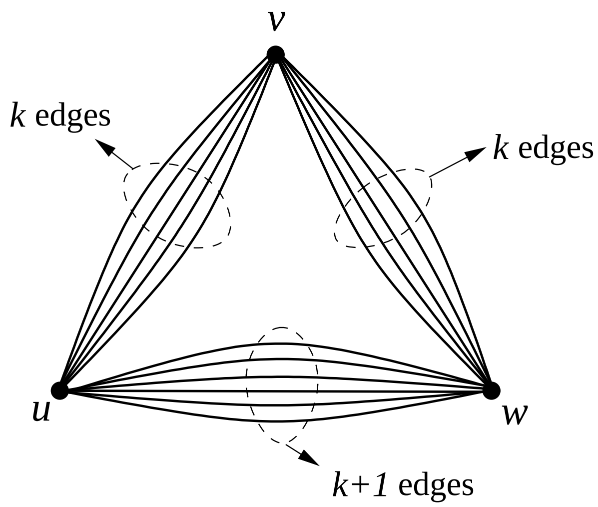


Figure 6.2: A graph with angular chromatic number $\lfloor \frac{3}{2}\Delta \rfloor$

there is no constant bound on the number of colours required in angular colourings of planar graphs, as the wheel graph on n vertices for instance (see Figure 6.1), requires n colours in any angular face colouring.

It is easy to see that for a graph with maximum degree Δ, we need at least Δ colours in any angular colouring. In fact there are planar graphs that require $\lfloor \frac{3}{2}\Delta \rfloor$ colours in any angular colouring. One of these graphs with $\Delta = 2k + 1$ is shown in Figure 6.2 (compare this graph with the graph of Figure 5.1). On the other hand, Ore and Plummer [43] proved that no planar graph requires more than 2Δ colours in any angular colouring. Thus, it is interesting to determine the best possible upper bound on the angular chromatic number of a planar graph with maximum degree Δ.

Angular colouring is equivalent to a vertex colouring problem, known as *cyclic colouring.* Consider a planar graph G and its dual G^*. An angular colouring of G is equivalent to a vertex colouring of G^*, such that two vertices receive different colours if they are incident with the same face; we call such a vertex colouring a cyclic colouring. The key parameter in G^*, which corresponds to $\Delta(G)$, is the maximum face size, denoted by Δ^*. The minimum number of colours required in any cyclic colouring of a planar graph G, denoted by $\chi_c(G)$, is the *cyclic* chromatic number of G. It is easy to see that in any cyclic

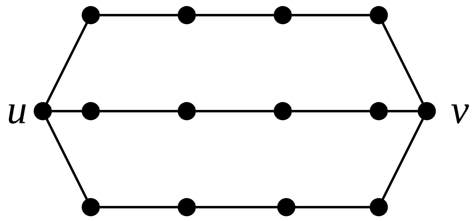


Figure 6.3: A graph with cyclic chromatic number $\lfloor \frac{3}{2}\Delta^* \rfloor$

colouring of the graph of Figure 6.3 all vertices should get different colours. Therefore, the cyclic chromatic number of this graph is $\lfloor \frac{3}{2}\Delta^* \rfloor$. In fact, this graph is the dual graph of the graph of Figure 6.2, with each path between u and v in Figure 6.3 corresponding to a set of parallel edges in the graph of Figure 6.2.

In the cyclic colouring of the dual graph G^* of a graph G, since we are colouring the vertices, we can ignore loops and multiple edges, or simply remove them to make G^* simple. According to [38] the following conjecture is implicitly stated by Borodin [11]:

Conjecture 6.2.1 *For every planar graph G with maximum face size Δ^*:*

$$\chi_c(G) \leq \lfloor \frac{3}{2}\Delta^* \rfloor.$$

It is not hard to see that this conjecture looks very similar to Wegner's conjecture on the chromatic number of the square of a planar graph. Not only do these two conjectures look similar, but also the known results on them are quite similar. The result of Ore and Plummer [43] provided a $2\Delta^*$ upper bound for $\chi_c(G)$. Borodin [12] improved this result to $2\Delta^* - 3$ for $\Delta^* \geq 8$. Then Borodin et al. [21] proved $\chi_c(G) \leq \lfloor \frac{9}{5}\Delta^* \rfloor$, and very recently, Sanders and Zhao [50] showed $\chi_c(G) \leq \lceil \frac{5}{3}\Delta^* \rceil$. The reducible configurations used in the proofs of the last two results are very similar to the reducible configurations used to prove the corresponding bounds for the chromatic number of the square of a planar graph, in [17, 16] and in Chapter 5. In fact our proofs in Chapter 5 were inspired by Sanders and Zhao [50].

Here we give a brief outline of the proof that $\chi_c(G) \leq \lfloor \frac{9}{5}\Delta^* \rfloor$. Consider an arbitrary planar graph G. Remember that the basic idea to prove $\chi(G^2) \leq \lceil \frac{9}{5}\Delta \rceil + 1$ was to show that there is a vertex v with $d_{G^2}(v) \leq \lceil \frac{9}{5}\Delta \rceil$. We have a similar approach here. Let us define the *cyclic degree* of a vertex v, denoted by $cd(v)$, to be the number of vertices, other than v, that are in the boundaries of the union of the faces containing v. The key reducible configuration in this proof is a vertex v with $cd(v) \leq \lfloor \frac{9}{5}\Delta^* \rfloor - 1$. The reducibility of this configuration follows from the fact that we can contract v on one of its neighbours to get a smaller planar graph G' with $\Delta^*(G') \leq \Delta^*(G)$, colour G' with $\lfloor \frac{9}{5}\Delta^*(G) \rfloor$ colours, and extend the colouring to v. We can prove the existence of this configuration in every planar graph using the Discharging Method. The following structure is the key in this proof: two faces f_1 and f_2, with a path $v_1 v_2 \ldots v_x$ of 2-vertices that belongs to the boundaries of both f_1 and f_2, i.e. f_1 and f_2 share this segment, and $x \geq \frac{\Delta^*}{5}$. If G has such a configuration then $cd(v_2) \leq |f_1| + |f_2| - x - 1 < \lfloor \frac{9}{5}\Delta^* \rfloor$, as wanted. We suggest that the reader takes a careful look back at the configuration described in Subsection 5.2.1 or the configuration in Figure 5.4, and compare it with the configuration described above to see their similar structure.

To prove $\chi_c(G) \leq \lceil \frac{5}{3}\Delta^* \rceil$ two main reducible configurations are required. One of them is a vertex v with $ch(v) \leq \lceil \frac{5}{3}\Delta^* \rceil - 1$. We call this reducible configuration, configuration $2'$ (as it corresponds to configuration 2 in Subsection 5.2.2). The other reducible configuration has a structure similar to that of configuration 3 in Subsection 5.2.2; so we call it configuration $3'$ (See [50] for a formal description of this configuration). These two configurations are the key configurations to prove $\chi_c(G) \leq \lceil \frac{5}{3}\Delta^* \rceil$. However, to improve this result asymptotically, we need to find a new reducible configuration, since there are planar graphs that are extremal for both configurations $2'$ and $3'$ in the following sense: every vertex v in these graphs has $cd(v) \geq \frac{5}{3}\Delta^* - c$ (for some constant c) and they do not have configuration $3'$. One of these graphs in shown in Figure 6.4. In this figure, every dashed line is a path of length $k-2$, and therefore, $\Delta^* = 3k+2$. Note that the structure of this graph is very similar to that of graph of Figure 5.11 (place a vertex in the center of each face of this graph and connect it to all the vertices on the boundary of

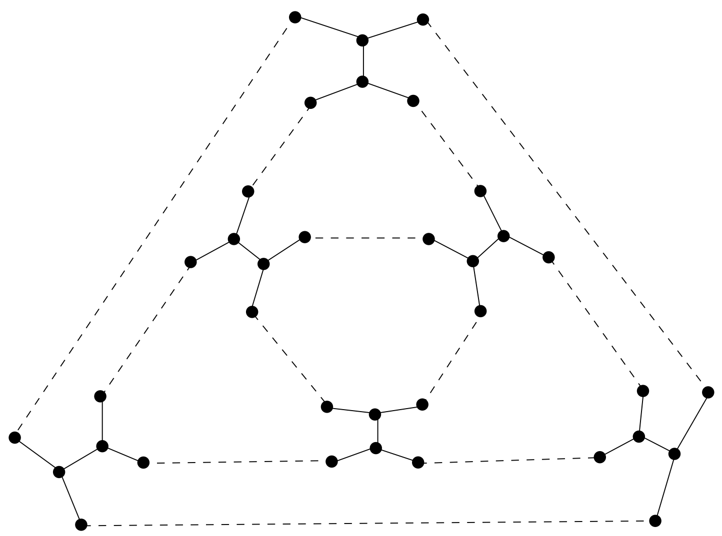


Figure 6.4: The extremal graph for configurations $2'$ and $3'$

that face). Similar to the discussion we had in the second paragraph of this section, to improve the bound $\chi_c(G) \leq \lceil \frac{5}{3}\Delta^* \rceil$ asymptotically (using the Discharging Method) we have to find a new reducible configuration (different from configurations $2'$ and $3'$), and if such a configuration exists, the graph of Figure 6.4 must have it. Therefore, the best way to find a new reducible configuration may be to look for it in the graph of Figure 6.4.

We think there is a correlation between these two problems in the following sense: any asymptotic improvement on the best known result on either of Conjectures 5.1.1 or 6.2.1 using the Discharging Method will require the introduction of a new reducible configuration. The structure of this new reducible configuration will probably help to find a new reducible configuration for the other problem and consequently to prove a similar asymptotic improvement. The reason backing this belief is a transformation from cyclic colouring to colouring the square of a planar graph, sketched below: Given a graph G, create G' by adding a new vertex v_f to each face f of G and connecting it to all the vertices in the boundary of f. Now the vertices in f have distance at most 2 from each other in G'. Therefore, any distance-2-colouring of G' yields a cyclic colouring of G. At first glance this transformation might seem as a correct reduction since the degree of every vertex $v_f \in G'$ is the same as the size of the corresponding face $f \in G$, and therefore, one might expect Δ^* in G to be the same as Δ in G. However, this is not necessarily true since there might be a vertex $v \in G$ with degree $d > \Delta^*$ and that vertex will have much larger degree than Δ^* in G'.

Although the transformation explained above is not a correct reduction from the cyclic colouring problem to the distance-2-colouring problem, it suggests that the former problem is easier than the latter. The following facts about the most recent results on these two problems support this guess: the most recent results for the distance-2-colouring problem on planar graphs were obtained using the ideas behind the reducible configurations used in the proofs of the corresponding results for the cyclic colouring problem (for example, as we said, the results of Chapter 5 were inspired by the work of Sanders and Zhao [50]). Furthermore, the structure of the reducible configurations used in the bounds for the distance-2-colouring problem, although similar to their counterparts for the cyclic colouring problem, are more complicated. Consequently, there are more discharging rules used in the proofs for the distance-2-colouring problem and these rules are more complicated. For example, the number of discharging rules in the results $\chi_c(G) \leq \lceil \frac{9}{5}\Delta^* \rceil$ (in [21]) and $\chi_c(G) \leq \lceil \frac{5}{3}\Delta^* \rceil$ (in [50]) are 7 and 7, whereas the number of discharging rules in the results $\chi(G^2) \leq \lceil \frac{9}{5}\Delta \rceil + 1$ (in [16, 17]) and $\chi(G^2) \leq \lceil \frac{5}{3}\Delta \rceil + O(1)$ (in Chapter 5) are 10 and 12, respectively.

Therefore, it might be better to first attack the cyclic coloring problem and improve the bound on the cyclic chromatic number of planar graphs asymptotically, and then possibly use the ideas of that proof to improve the bound on the chromatic number of the square of planar graphs.

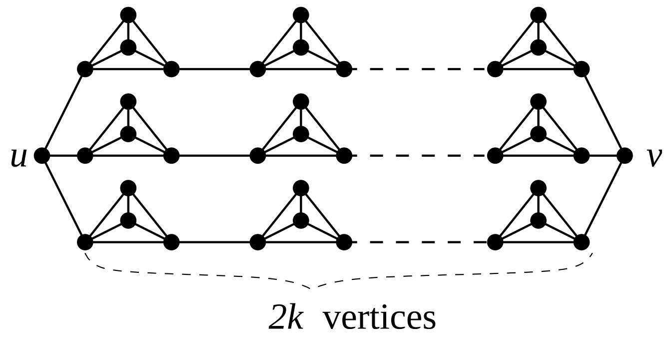


Figure 6.5: A graph with minimum degree 3 and high cyclic chromatic number

6.2.2 Distance-2-Colouring in Planar Graphs With High Connectivity

Consider the cyclic colouring problem. In the previous subsection we saw that there are planar graphs, such as the one in Figure 6.3, whose cyclic chromatic number has asymptotic order of $\frac{3}{2}\Delta^*$. But this graph is not 3-connected and has many vertices of degree 2. What if we assume that the graph is 3-connected? For this case, i.e. for 3-connected planar graphs, Plummer and Toft [44] conjectured that the number of colours required in a cyclic colouring is at most $\Delta^* + 2$:

Conjecture 6.2.2 *[44] For every 3-connected planar graph G with maximum face size Δ^*: $\chi_c(G) \leq \Delta^* + 2$.*

Note that having only minimum degree at least 3 instead of 3-connectivity is not sufficient to prove the upper bound $\chi_c(G) \leq \Delta^* + O(1)$. For instance, in the graph of Figure 6.5 (which is a modification of graph of Figure 6.3), $\delta = 3$, $\Delta^* = 5k+2$, and $\chi_c(G) \geq 6k+2$. However, neither this graph nor the graph of Figure 6.3 is 3-connected.

Plummer and Toft [44] proved that for 3-connected planar graphs $\chi_c(G) \leq \Delta^*+9$ and that $\chi_c(G) \leq \Delta^*+4$ if $\Delta^* \geq 42$. Borodin and Woodall [10] and Horňák and Jendrol' [37] proved Conjecture 6.2.2 when $\Delta^* \geq 61$ and $\Delta^* \geq 24$, respectively. Furthermore, Borodin and Woodall [10] and Enomoto et al. [25] showed that the cyclic chromatic number of 3-connected planar graphs is at most $\Delta^* + 1$ if $\Delta^* \geq 122$ and $\Delta^* \geq 60$, respectively.

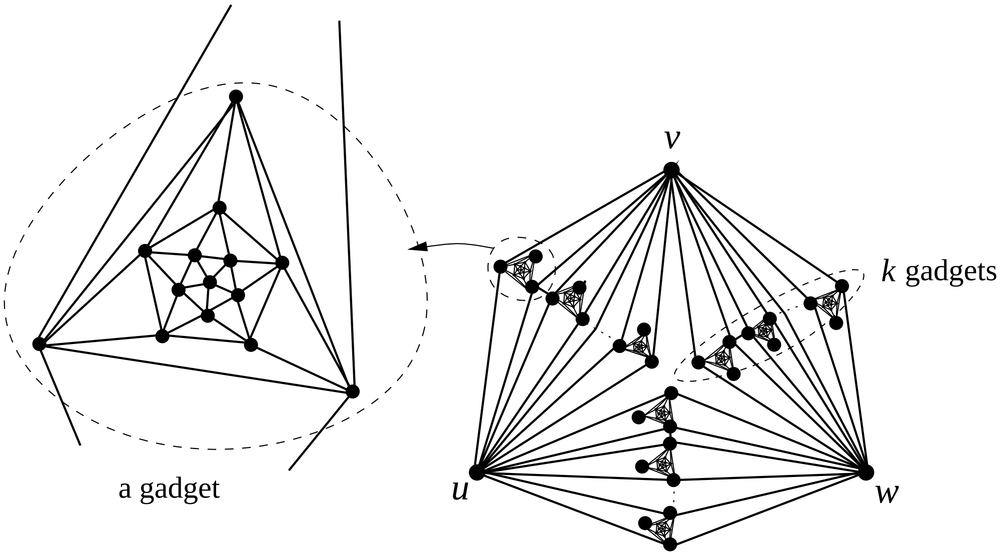


Figure 6.6: Graph G with minimum degree 5 and $\chi(G^2) \geq \frac{3}{2}\Delta$

Under some similar restrictions, can we have a similar upper bound (in which the coefficient of Δ is 1) for the distance-2-colouring problem? It is natural to ask:

> **Question:** If G is a planar graph with high connectivity (say at least 4- or 5-connected) then can we prove $\chi(G^2) \leq \Delta + O(1)$?

Note that having only high minimum degree instead of high connectivity is not sufficient to prove the upper bound $\chi(G^2) \leq \Delta + O(1)$ or even to bring the multiplicative constant below $\frac{3}{2}$. For instance, we can modify the graph of Figure 5.1 (in a similar manner to the way we modified the graph of Figure 6.3) and obtain the graph of Figure 6.6. In this graph, v is adjacent to both u and w, each of u, v, w is connected to $2k$ gadgets as shown on the left side of the figure, $d(u) = d(w) = 4k+1$, $d(v) = \Delta = 4k+2$, and $\delta = 5$. Since u, v, w, and all their neighbours are at distance at most two from each other, all of them must get different colours in any distance-2-colouring. Thus $\chi(G^2) \geq 6k+3 = \frac{3}{2}\Delta$. So for this graph, which has minimum degree 5, not only is $\chi(G^2)$ not $\Delta + O(1)$, it actually has the same asymptotic order as that of the extremal graph of Figure 5.1. In fact, if we modify the graph of Figure 5.1 slightly so that all u, v, w have degree $2k$, then

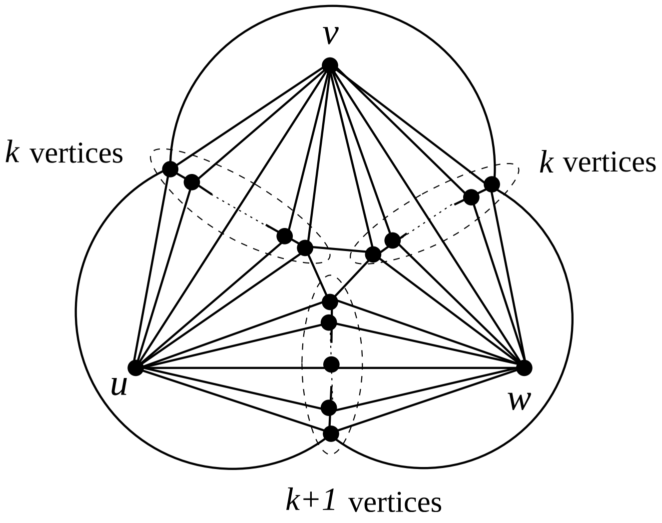


Figure 6.7: A 3-connected graph G with $\chi(G^2) = \frac{3}{2}\Delta + 1$

we obtain a graph, which is a subgraph of G (in Figure 6.6) and has the same maximum degree as G.

The assumption that the given planar graph is 3-connected is not sufficient either since we can modify the graph of Figure 5.1 such that it becomes 3-connected without changing Δ, by adding an edge between every two consecutive neighbours of u in clockwise order, and similarly between every two consecutive neighbours of v and w (See Figure 6.7).

The suitable assumption for this problem might be 4-connectivity. This assumption immediately rules out the extremal graphs of Figure 6.6 and 6.7. But we don't know if it actually helps to reduce the coefficient of Δ down to 1 (or even below $\frac{3}{2}$). This problem does not seem to be studied in the literature. It would be very interesting if with this extra condition we could match the results of cyclic colouring of 3-connected planar graphs.

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Appendix A

More Hand-checkable Proofs For Theorem 3.1.1

In Section 3.3.2 we listed 15 reducible configurations required in the proof of Theorem 3.1.1 and provided hand-checkable proofs of the first 7 ones. In this appendix, we explain the hand-checkable proofs of configurations 8 to 12. All these proofs have a very similar pattern; similar to the proofs of configurations 4-7 that we saw in Section 3.3.2. The author has also proved, by hand, that the 49 subconfigurations for configurations 13-15 are reducible, but including the proofs here would make this section too long and too repetitive (even more so than it is now!). These missing proofs follow the same patterns as the included proofs, and armed with this plethora of examples, it will be very easy (and time-consuming) for the reader to generate any of the missing proofs that he/she desires.

Proof of configuration 8: Instead, we prove that the four configurations shown in Figures A.1(a), (b), (c), and (d), are reducible. Each of these configurations contains a semi-simple face f_1, in which the both neighbours of its type 1 vertex, which are not incident with f_1, are 3-vertices. Note that any configuration that contains two semi-simple faces that share a type 1 vertex must have one of the configurations in Figure

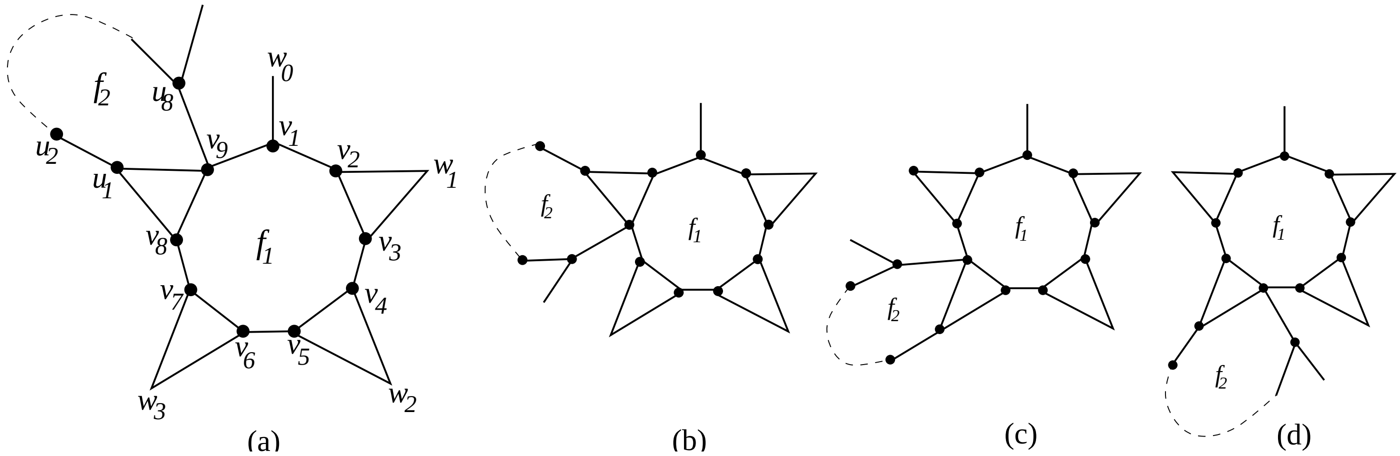
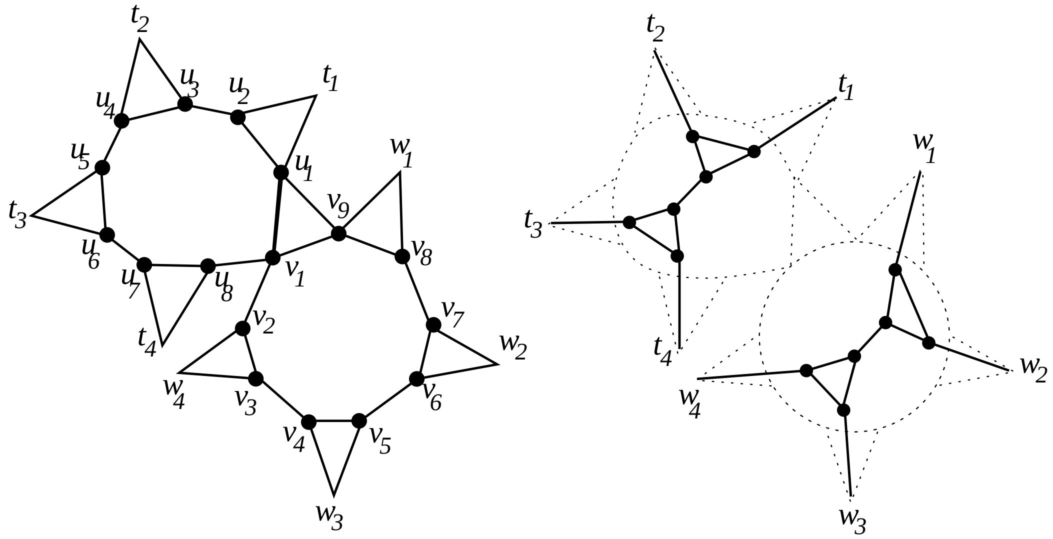


Figure A.1: Two semi-simple faces sharing their type 1 vertex

A.1. We first give the proof for the configuration of Figure A.1(a): By minimality of G, there is a 3-colouring of $G' = G - v_1v_2$, called C. So $C(v_1) = C(v_2)$, which we can assume is equal to 1. Consider this colouring induced on G. By the chaining argument $C(v_4) = C(v_6) = C(v_8) = C(u_2) = 1$, otherwise we could 3-colour G. Without loss of generality, assume $C(w_0) = 2$. So $C(v_9) = 3$ and $C(u_1) = 2$, otherwise we could set $C(v_1) = 3$. If $C(u_8) = 1$ then we could exchange $C(u_1)$ with $C(v_9)$ and set $C(v_1) = 3$. Therefore $C(u_8) = 2$. Now set $C(v_1) = 3$, $C(v_9) = 1$ and assign a colour different from 1 and $C(v_7)$ (which is either 2 or 3) to v_8 and give a colour different from $C(v_9)$ (which is 1) and $C(v_8)$ to u_1 (we can do this because $C(v_9) = C(u_2) = 1$). This gives a 3-colouring of G, which is a contradiction.

Using very similar arguments, we can show that the configurations of Figures A.1(b), (c), and (d) are reducible. ■

Proof of configuration 9: Suppose that f_1 and f_2 are two semi-type 2 faces sharing a type 1 vertex. There are eight possible configurations of this type up to isomorphism, we consider each one separately. Assume that $v_1, \ldots, v_9$ are the vertices of f_1, where v_9 is the type 2 vertex. In the first two cases we assume that v_1 is the type 1 vertex of f_1 (Figures A.2(a) and A.3(a)). The other cases are based on v_2, v_3, or v_4 being the type 1 vertex of f_1, shown in Figure A.4.



(a) (b)

Figure A.2: Two semi-type 2 faces sharing their type 1 vertex

Configuration of Figure A.2(a): In this case u_1 is the type 2 vertex of f_2. First we remove some vertices and edges and add two gadgets each similar to the one in lemma 4. The vertices to be removed are $v_1, \dots, v_9$ and $u_1, \dots, u_8$, and the new graph G' after adding the gadgets is shown in Figure A.2(b). It is straightforward to verify that: (i) $G' \in \mathcal{G}_8$, (ii) because of minimality of G there is a 3-colouring of G', say C, and (iii) $w_1, \dots, w_4$ cannot all have the same colour in C. Also, $t_1, \dots, t_4$ cannot all have the same colour in C.

Consider this 3-colouring induced on G. First we show that $C(w_1) \neq C(t_1)$. By contradiction, assume that $C(w_1) = C(t_1) = 3$. Now we can extend C to a new colouring C' in this way: for all common vertices of G and G', C' and C are equal. Then assign $C'(v_1) = 3$, and colour $u_8, u_7, \dots, u_1$ greedily. Note that by the time we reach to u_1 it has three coloured neighbours but two of them (v_1 and t_1) have the same colour. Assume that $C'(u_1) = 2$. Set $C'(v_9) = 1$, $C'(v_8) = 2$, and colour $v_2, v_3, \dots, v_6$ greedily. Finally, assign a colour different from $C'(v_6)$ and $C'(w_2)$ to v_7. By minimality of G, both v_7 and v_8 have the same colour, which is 2. By the chaining argument we must have $C'(v_5) = C'(v_3) = C'(v_1) = 2$, but $C'(v_1) = 3$. This contradiction shows that

$C(w_1) \neq C(t_1)$.

Now we extend C to colour the uncoloured vertices of G in a different way. Assume that $C(w_1) = 3$. Since $C(t_1) \neq C(w_1)$ we can assign $C(u_1) = 3$ and colour the uncoloured vertices of G greedily in the following order: $u_2, \ldots, u_8$, $v_1, v_9, v_8, v_2, v_3, \ldots, v_6$. Note that by the time we want to colour v_9 there are two neighbours of it (u_1 and w_1) that have the same colour and so we can find a colour for v_9. We also assign a colour different from $C(v_6)$ and $C(w_2)$ to v_7. By definition of G, $C(v_8) = C(v_7)$, which we can assume is equal to 1, By the chaining argument $C(v_5) = C(v_3) = C(v_1) = 1$, and so $C(v_9) = 2$.

Suppose that $C(u_8) \neq 2$. We can set $C(v_1) = 2$, $C(v_9) = 1$, and $C(v_8) = 2$, unless $C(v_2) = 2$ and by the chaining argument $C(v_2) = C(v_4) = C(v_6) = 2$. But this means that all $w_1, \ldots, w_4$ have colour 3, which contradicts property (iii).

Now assume that $C(u_8) = 2$. If we could exchange $C(u_8)$ and $C(u_7)$ then $C(u_8)$ becomes different from 2 and we can use the argument of the previous paragraph. This shows that $C(u_6) = 2$ and by the chaining argument $C(u_4) = C(u_2) = 2$. If $C(u_3) \neq 3$ then we can modify C in the following way: set $C(u_2) = 3$, $C(u_1) = 2$, $C(v_1) = 3$, $C(v_9) = 1$, $C(v_8) = 2$, exchange $C(v_2)$ with $C(v_3)$ if $C(v_2) = 3$, exchange $C(v_4)$ with $C(v_5)$ if $C(v_4) = 3$, and finally exchange $C(v_6)$ with $C(v_7)$ if $C(v_6) = 3$, which yields a 3-colouring of G. Therefore, $C(u_3) = 3$ and by the chaining argument $C(u_5) = C(u_7) = 3$. But this means that all $t_1, \ldots, t_4$ have colour 1, again contradicting (iii).

Configuration of Figure A.3(a): In this case u_1 is a 3-vertex in f_2. First we remove $v_2, \ldots, v_8$ and add a gadget similar to that of Lemma 4. The new graph G' is shown in Figure A.3(b). It can be easily shown that: (i) $G' \in \mathcal{G}_8$, (ii) because of minimality of G there is a 3-colouring of G', say C, and (iii) $w_1, \ldots, w_4$ cannot all have the same colour in C.

Consider this 3-colouring induced on G. We extend C by colouring the uncoloured vertices of G greedily in the following order: $v_8, v_2, \ldots, v_6$. Then assign a colour different from $C(v_6)$ and $C(w_2)$ to v_7. By minimality of G, $C(v_7) = C(v_8)$ which we can assume

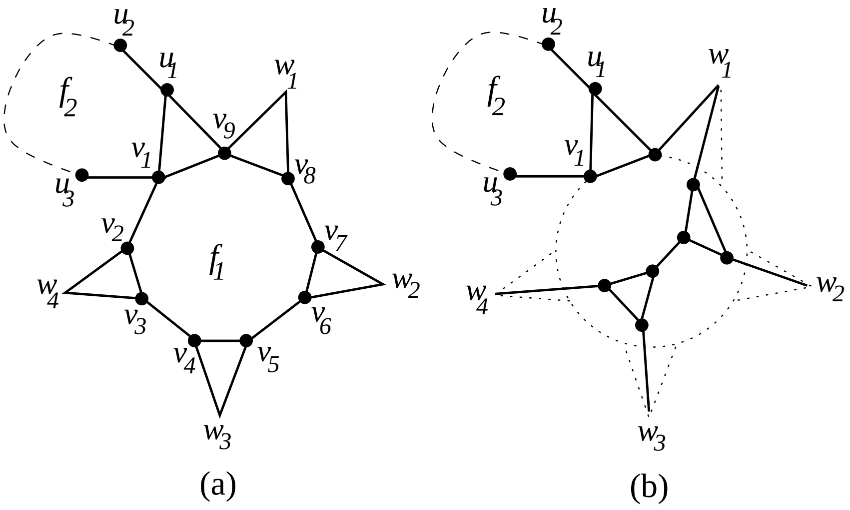


Figure A.3: Two semi-type 2 faces sharing their type 1 vertex

both are 1. By the chaining argument $C(v_5) = C(v_3) = C(v_1) = 1$. Without loss of generality, assume that $C(v_9) = 2$ and so $C(u_1) = C(w_1) = 3$.

If $C(u_3) = 3$ then we could set $C(v_8) = 2$, $C(v_9) = 1$, $C(v_1) = 2$, then exchange $C(v_2)$ with $C(v_3)$ if $C(v_2) = 2$, and then exchange $C(v_5)$ with $C(v_4)$ if $C(v_4) = 2$. In this case $C(v_6) \neq 2$, otherwise $w_1, \ldots, w_4$ all are coloured 3, a contradiction.

So assume that $C(u_3) = 2$. If $C(u_2) = 2$ then we can exchange $C(v_1)$ with $C(u_1)$, $C(v_2)$ with $C(v_3)$, $C(v_4)$ with $C(v_5)$, and $C(v_6)$ with $C(v_7)$, which gives a 3-colouring of G. If $C(u_2) = 1$ then we set $C(u_1) = 2$, $C(v_1) = 3$, $C(v_9) = 1$, and $C(v_8) = 2$. Then we can exchange $C(v_2)$ with $C(v_3)$ if $C(v_2) = 3$, then exchange $C(v_4)$ with $C(v_5)$ if $C(v_4) = 3$, and finally exchange $C(v_6)$ with $C(v_7)$ if $C(v_6) = 3$. So we get a 3-colouring of G, which again is a contradiction.

Configurations of Figure A.4: The other possibilities, up to isomorphism, for two semi-type 2 faces to share their type 1 vertex are shown in Figure A.4. Here we only give the proof for configuration of Figure A.4(A). The proof for the other configurations is almost the same.

By minimality of G, there is a 3-colouring of $G - (v_7, v_8)$, called C. Consider this colouring induced on G in which both v_7 and v_8 have the same colour. Without loss of

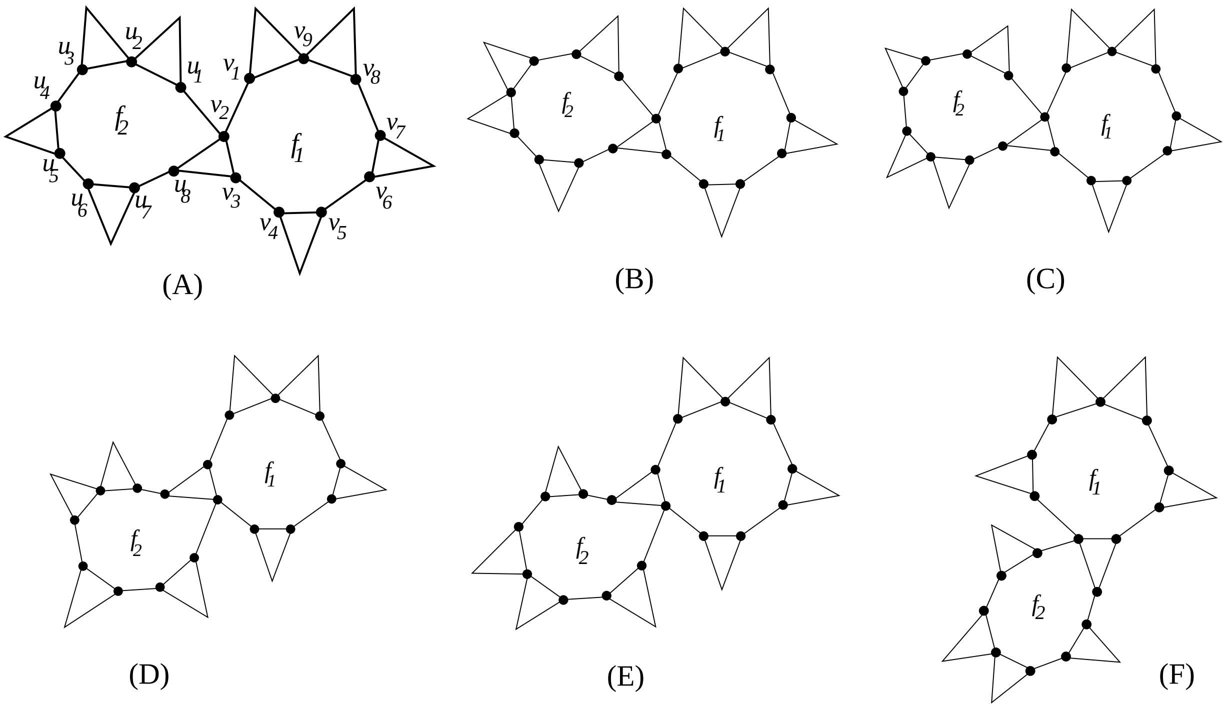


Figure A.4: Two semi-type 2 faces sharing their type 1 vertex

generality, assume that $C(v_7) = C(v_8) = 1$. By the chaining argument $C(v_5) = C(v_3) = C(u_7) = C(u_5) = C(u_3) = 1$. So $C(v_2) \neq 1$.

First assume that both u_1 and v_1 have the same colour different from 1, say 2. Then we can exchange $C(v_2)$ with $C(v_3)$, $C(v_4)$ with $C(v_5)$, and $C(v_6)$ with $C(v_7)$, which yields a 3-colouring of G, a contradiction. Also, $\{C(v_1), C(u_1)\} \neq \{2, 3\}$, since $C(v_2) \neq 1$. So at least one of $C(v_1)$ or $C(u_1)$ is 1.

Assume that $C(v_1) = 1$ and $C(u_1) = 2$. So $C(v_2) = 3$. If $C(v_9) = 2$ we can set $C(v_1) = C(v_8) = 2$ and $C(v_9) = 1$ which gives a 3-colouring of G. On the other hand, if $C(v_9) = 3$ we can modify C in this way: set $C(v_2) = 1$, $C(v_1) = 3$, $C(v_9) = 1$, $C(v_8) = 3$, assign a colour different from $C(v_4)$ and 1 to v_3. Now since $C(v_2) = C(u_7) = 1$, we can assign a colour different from 1 and $C(v_3)$ to u_8. This gives a 3-colouring of G, an obvious contradiction.

Now, let's assume that $C(u_1) = 1$ and $C(v_1) = 2$. So $C(v_2) = 3$ and $C(u_8) = 2$. If $C(u_2) = 2$ then set $C(u_1) = 2$, $C(u_2) = 1$, $C(u_3) = 2$, exchange $C(u_4)$ with $C(u_5)$, $C(u_6)$

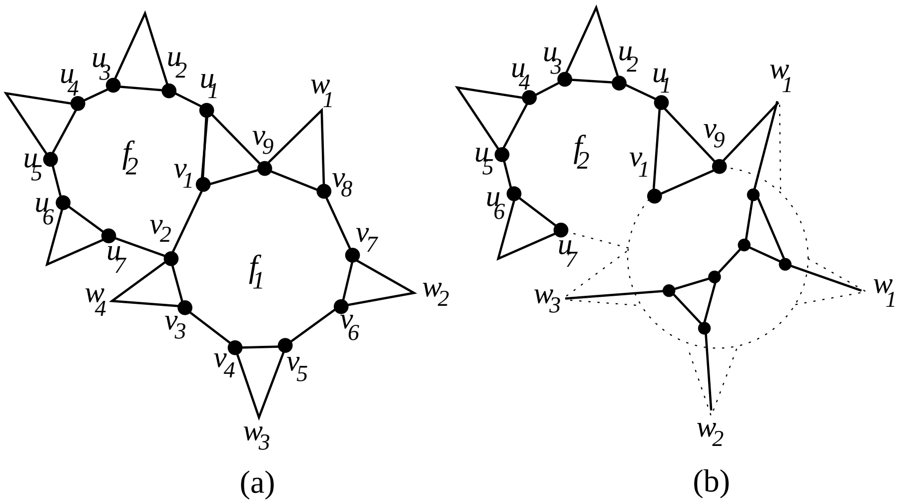


Figure A.5: A semi-type 2 face sharing a type 1 vertex with a type 1 face

with $C(u_7)$, $C(u_8)$ with $C(v_3)$, $C(v_4)$ with $C(v_5)$, and $C(v_6)$ with $C(v_7)$, which yields a 3-colouring of G. If $C(u_2) = 3$ then set $C(u_1) = C(u_3) = 3$, $C(u_2) = 1$, $C(v_2) = 1$, exchange $C(u_4)$ with $C(u_5)$, and $C(u_6)$ with $C(u_7)$. Assign a colour different from $C(v_2)$ (which is 1) and $C(u_7)$ to u_8. Then assign a colour different from 1 and $C(u_8)$ to v_3. Now exchange $C(v_4)$ with $C(v_5)$ and $C(v_6)$ with $C(v_7)$. This again is a 3-colouring of G.

Finally, assume that $C(v_1) = C(u_1) = 1$. Without loss of generality, assume that $C(v_9) = 2$. If $C(v_2) = 2$ we exchange it with $C(u_8)$ so that $C(v_2) \neq C(v_9)$. Now set $C(v_1) = 2$, $C(v_9) = 1$, and $C(v_8) = 2$. This yields a 3-colouring of G, which is a contradiction. ∎

Proof of configuration 10: There are four possible configurations of this type up to isomorphism, shown in Figures A.5(a), A.6(A1), A.6(B1), and A.6(C1). We consider each one separately:

Configuration of Figure A.5(a): First remove $v_2, v_3, \ldots, v_8$ and all the incident edges and create the graph G' as in Figure A.5(b) by adding a gadget. It is straightforward to verify that: (i) $G' \in \mathcal{G}_8$, (ii) because of minimality of G there is a 3-colouring of G', say C, and (iii) $w_1, \ldots, w_4$ cannot all have the same colour in C.

Consider this 3-colouring induced on G. We extend C by colouring the uncoloured vertices of G greedily in the following order: $v_2, v_8, v_7, \ldots, v_4$. We also assign a colour different from $C(v_2)$ and $C(w_4)$ to v_3. By definition of G, $C(v_3) = C(v_4)$, which we can assume is equal to 1, and by the chaining argument $C(v_6) = C(v_8) = 1$ and at least one of $C(v_1)$ or $C(u_7)$ must be 1.

First assume that $C(u_7) = 1$ and $C(v_1) \neq 1$. By the chaining argument $C(u_5) = C(u_3) = C(u_1) = 1$. Without loss of generality assume that $C(v_9) = 2$ and so $C(w_1) = 3$. Now set $C(v_9) = 1$ and $C(u_1) = C(v_8) = 2$, exchange $C(u_2)$ with $C(u_3)$, $C(u_4)$ with $C(u_5)$, $C(u_6)$ with $C(u_7)$, and $C(v_3)$ with $C(v_2)$. The only conflict we may have is between $C(v_8)$ and $C(v_7)$, which happens if $C(v_7) = 2$. We can exchange $C(v_7)$ with $C(v_6)$, unless $C(v_5) = 2$. In this case we can exchange $C(v_5)$ with $C(v_4)$, unless $C(v_3) = 2$. But this means that all $w_1, \ldots, w_4$ have been coloured 3, which contradicts (iii).

Now assume that $C(v_1) = 1$ and $C(u_7) \neq 1$. By the chaining argument $C(u_2) = C(u_4) = C(u_6) = 1$. Assume that $C(v_9) = 2$. Set $C(v_9) = 1$, $C(v_1) = C(v_8) = 2$, and exchange $C(v_2)$ with $C(v_3)$. Similar to the previous case we can solve the possible conflict between $C(v_8)$ and $C(v_7)$, unless all $w_1, \ldots, w_4$ have colour 3, which is impossible, according to (iii).

Finally, assume that $C(v_1) = C(u_7) = 1$. If we could modify $C(v_1)$ or $C(u_7)$ then we would reduce to the one of the two cases we just considered. Therefore, by the chaining argument and starting from u_7: $C(u_5) = C(u_3) = C(u_1) = 1$, which is impossible, since $C(v_1) = 1$. This completes the proof of this configuration.

The other three possible configuration of this kind, up to isomorphism, are shown in Figure A.6(A1), (B1), and (C1). First consider the configuration of Figure A.6(A1).

Remove $v_1, v_2, \ldots, v_9$ and $u_1, \ldots, u_7$ and all the incident edges and create the graph G' as in Figure A.6(A2). It is straightforward to verify that: (i) $G' \in \mathcal{G}_8$, (ii) because of minimality of G there is a 3-colouring of G', say C, and (iii) $w_1, \ldots, w_6$ cannot all have the same colour in C.

Consider this 3-colouring induced on G. We extend C by colouring the uncoloured

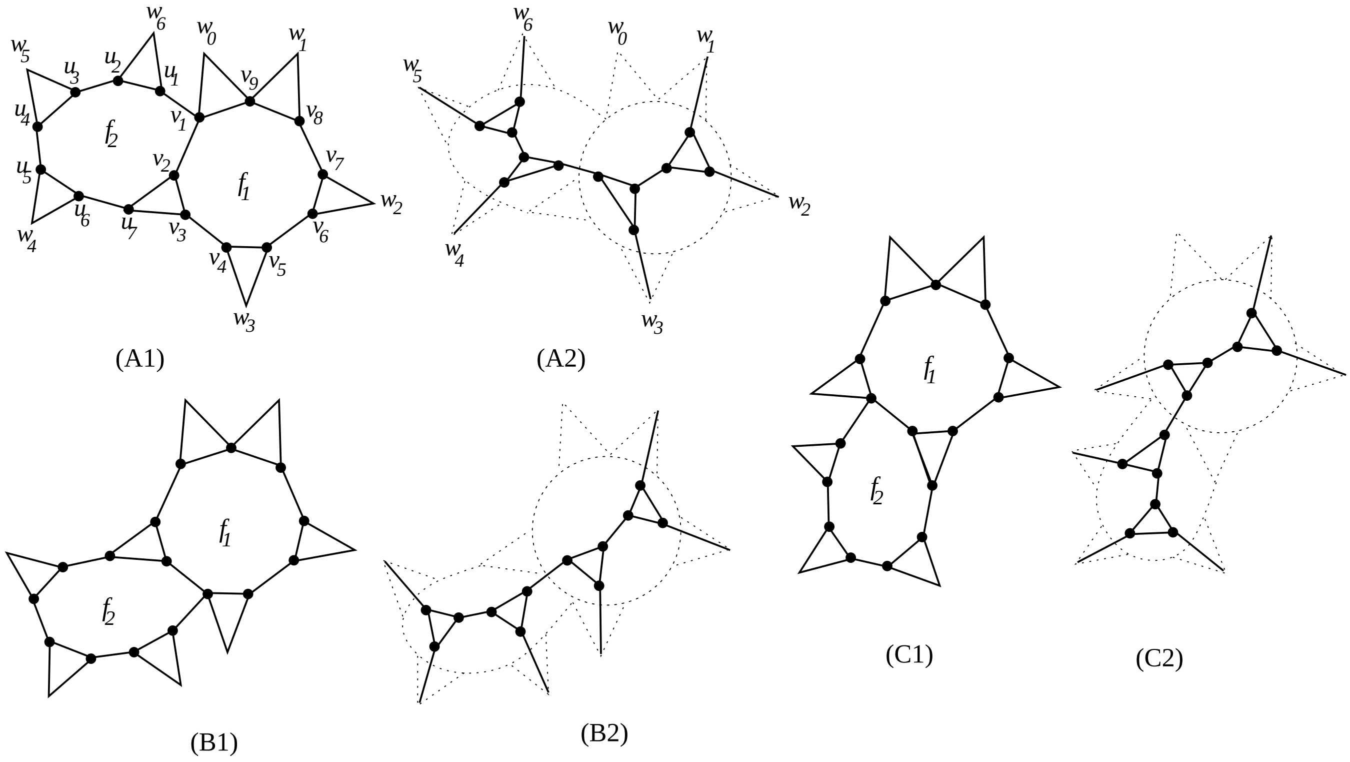


Figure A.6: A semi-type 2 face sharing a type 1 vertex with a type 1 face

vertices of G greedily in the following order: $v_9, v_1, v_8, v_7, \ldots, v_3, u_1, u_2, \ldots, u_7$. We also assign a colour different from $C(v_3)$ and $C(u_7)$ to v_2. By definition of G, $C(v_1) = C(v_2)$, which we can assume is equal to 1, and by the chaining argument $C(v_4) = C(v_6) = C(v_8) = C(u_6) = C(u_4) = C(u_2) = 1$. Without loss of generality assume that $C(v_9) = 2$.

If $C(u_1) = 3$ then we can set $C(v_1) = C(v_8) = 2$, $C(v_9) = 1$, then exchange $C(v_7)$ with $C(v_6)$ if $C(v_7) = 2$, then exchange $C(v_5)$ with $C(v_4)$ if $C(v_5) = 2$, and finally exchange $C(v_3)$ with $C(u_7)$ if $C(v_3) = 2$. This yields a 3-colouring of G.

So we can assume that $C(u_1) = 2$. If we could exchange $C(u_1)$ with $C(u_2)$ we could use the argument of the previous paragraph. So by the chaining argument $C(u_3) = C(u_5) = 2$. We could assign $C(v_1) = C(v_8) = 2$, $C(v_9) = 1$, exchange $C(u_1)$ with $C(u_2)$, $C(u_3)$ with $C(u_4)$, $C(u_5)$ with $C(u_6)$, and exchange $C(v_7)$ with $C(v_6)$ if $C(v_7) = 2$, unless $C(v_5) = 2$. This means that all $w_1, \ldots, w_6$ have been coloured 3 in C, contradicting property (iii) we just mentioned. This completes the proof of this configuration.

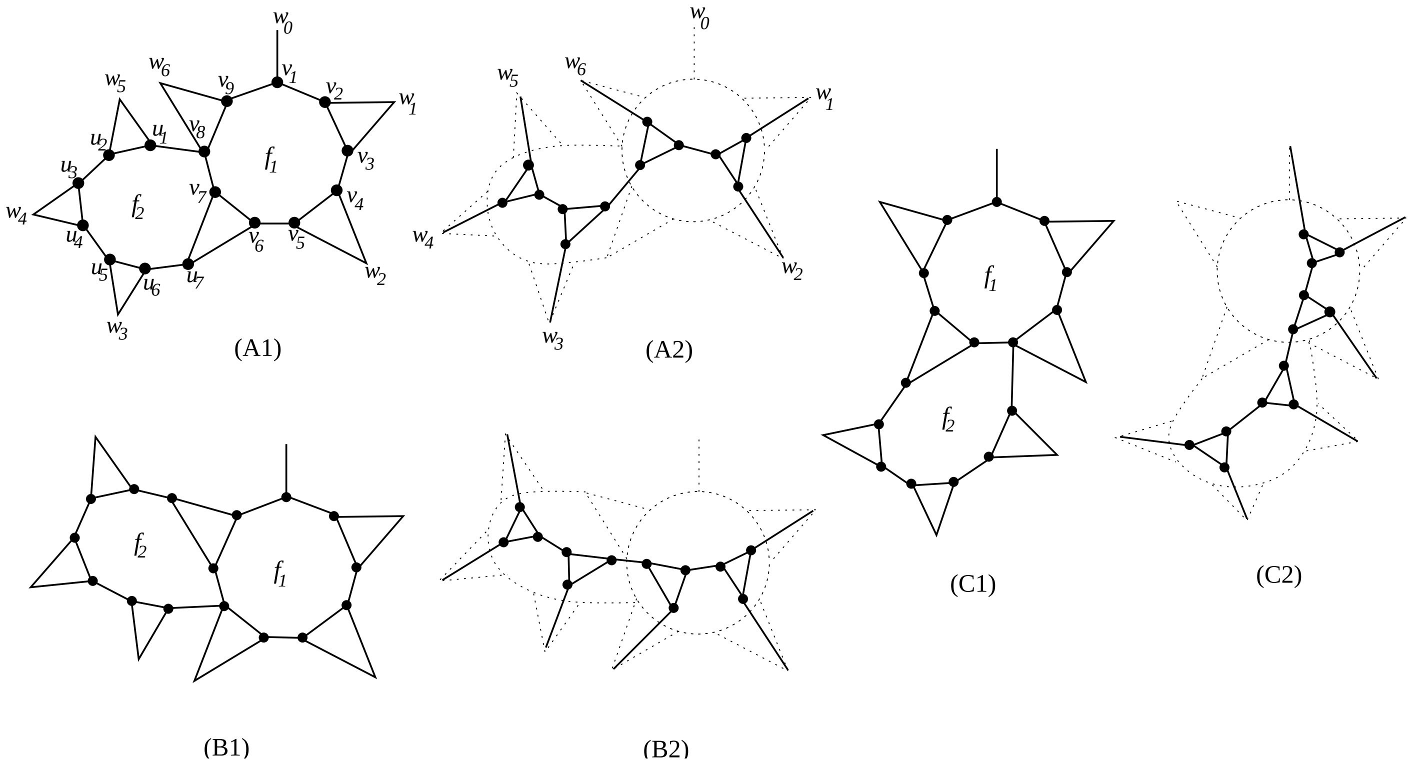


Figure A.7: A semi-simple face sharing a type 1 vertex with a type 1 face

Using a very similar argument, we can prove the reducibility of configurations of Figure A.6(B1) and (C1). The gadget we have to add in each case is shown in Figures A.6(B2) and (C2), respectively.

■

Proof of configuration 11: It is straightforward to check that there are five possible configurations of this type up to isomorphism. One of them is the same as the configuration of Figure A.3(a), and the other four ones are equivalent to the configurations of Figures A.1(A1), A.1(B1), A.1(C1), and A.1(D1). Each of these configurations are already proved to be reducible. ■

Proof of configuration 12: There are three possible configuration up to isomorphism, shown in Figure A.7(A1), (B1), and (C1). Let's consider (A1).

First remove $v_1, \ldots, v_9$ and $u_1, \ldots, u_7$, and all the incident edges and create the graph G' as in Figure A.7(A2). It is straightforward to verify that: (i) $G' \in \mathcal{G}_8$, (ii) because of minimality of G there is a 3-colouring of G', say C, and (iii) $w_1, \ldots, w_6$ cannot all have the same colour in C.

Consider this colouring induced on G and extend it by colouring the uncoloured vertices of G in the following order: $v_1, v_9, v_8, u_1, \ldots, u_7, v_7, v_6, \ldots, v_3$ Also, assign a colour different from $C(v_3)$ and $C(w_1)$ to $C(v_2)$. By minimality of G, $C(v_1) = C(v_2)$, which we can assume is 1. By the chaining argument $C(v_4) = C(v_6) = C(u_6) = C(u_4) = C(u_2) = C(v_8) = 1$. Without loss of generality assume that $C(w_0) = 2$. So $C(v_9) = 3$, otherwise we could set $C(v_1) = 3$. Note that we can safely exchange $C(v_7)$ with $C(u_7)$. If $C(u_1) \neq 3$ we can exchange $C(v_9)$ with $C(v_8)$ and set $C(v_1) = 3$. So $C(u_1) = 3$ and by the chaining argument $C(u_3) = C(u_5) = C(v_5) = C(v_3) = 3$. But this means that all $w_1, \ldots, w_6$ have colour 3, contradicting property (iii).

Using a very similar argument, we can prove the reducibility of configurations of Figures A.7(B1) and (C1). The gadget we have to add in each case is shown in parts (B2) and (C2), respectively. ■

The proofs of reducibility of configurations 13, 14, and 15 follow very similar steps. We omit the hand-checkable proofs of them.

Appendix B

The C Program used in Chapter 3

This program and the file containing the reducible configurations and the description of the program is also available at `ftp://ftp.cs.toronto.edu/csrg-technical-reports/458/`.

```
/* Version 1.1,  July 2002 */

#include <stdio.h>
#include <stdlib.h>
#include <string.h>
#include <time.h>

#define Max_No_of_vertices 50
#define Error_filename "UnColorable_Config.txt"

int Nvertices, Nedges,  /* No. of vertices and edges of the configuration */
  Nbound,               /* No. of boundary neighbors */
  NConstrained_groups,  /* No. of Constrained groups */
  Ncolored,             /* No. of colored vertices so far */
```

```
  Nof_colorings,        /* No. of differenet colorings found for a config. */
  is_in_bound [Max_No_of_vertices],
                        /* is_in_bound [v] = 1 if v is a boundary
                                              neighbor, 0 otherwise */
  adj_list[Max_No_of_vertices][Max_No_of_vertices],
                        /* The adjacancy list; for vertex v adj_list[v][0]
                                              specifies the degree of v */

  bound[Max_No_of_vertices],       /* The list of boundary neighbors */
  non_bound[Max_No_of_vertices],   /* The list of non-boundary vertices */

  constrained_groups[10][Max_No_of_vertices],
      /* The list of constrained groups; the vertices in a group are those
         boundary neighbors which must not all have the same color,
         enforced by a gadget. For group i constrained_groups[i][0]
         specifies the number of vertices in that group */

  color[Max_No_of_vertices],  /* Color of vertex v is color[v], 0 if
                                                   it is not colored */
  Nof_configurations,         /* No. of configuration in the file */
  current_conf;  /* index of the current configuration being tested */
  FILE *fErrors; /* The file to write in any non-reducible configuration */

/***************************************************************************/
/* Function Prototypes */
int Check_Boundary_Colorings (int NColored_bound);
void Read_Data (char *filename);
```

```
void UnColorable (void);
int Check_Extendable (int vertex);
int Valid_Boundary_Coloring (void);
int Check_Boundary_Colorings (int NColored_bound);


/**********************************************************************/
/* Read the configurations from a file whose name is "filename",
                        one by one, and check reducibility of each  */
void Read_Data (char *filename){


  FILE *fin;
  int i, j, v1, v2, tempvertex;
  char tmpStr[100];


  /* Openning the input file */
  if ((fin = fopen (filename, "r")) == NULL) {
    printf ("Cannot open the input file! \n");
    exit (1);
  }
  /* Openning the output (i.e. error) file */
  if ((fErrors = fopen (Error_filename, "w")) == NULL) {
    printf ("Cannot open the output file! \n");
    fclose(fin);
    exit (1);
  }
  fscanf (fin, "%d \n", &Nof_configurations);
```

```
/* Reading the information of configurations one by one and
                              checking the reducibility of them */
current_conf = 1;
for (current_conf = 1; current_conf <= Nof_configurations; current_conf++){
  fgets (tmpStr, sizeof (tmpStr), fin);
  fscanf (fin, "%d %d \n", &Nvertices, &Nedges);
  printf ("%d %d \n", Nvertices, Nedges);
  Ncolored = 0;
  Nbound = 0;
  Nof_colorings = 0;
  for (i = 1; i <= Nvertices; i++){
    adj_list[i][0] = 0;
    color[i] = 0;
    is_in_bound[i] = 0;
  }

  /* Reading the adjacancy lists of the current configuration    */
  for (i = 1; i <= Nedges; i++){
    fscanf (fin, "%d %d \n", &v1, &v2);
    adj_list[v1][++adj_list[v1][0]] = v2;
    adj_list[v2][++adj_list[v2][0]] = v1;
  }

  /* Setting up the boundary neighbors */
  j = 0;
  for (i = 1; i <= Nvertices; i++){
    if (adj_list[i][0] <= 2){
```

```
      bound[++Nbound] = i;
      is_in_bound[i] = 1;
    }
    else non_bound[++j] = i;
  }

  i=1;
  while (is_in_bound[adj_list[non_bound[1]][i]]) i++;
  tempvertex = adj_list[non_bound[1]][i];
  adj_list[non_bound[1]][i]=adj_list[non_bound[1]][adj_list[non_bound[1]][0]];
  adj_list[non_bound[1]][adj_list[non_bound[1]][0]]=tempvertex;
  j=1;
  while (adj_list[tempvertex][j]!=non_bound[1]) j++;
  adj_list[tempvertex][j]=adj_list[tempvertex][adj_list[tempvertex][0]];
  adj_list[tempvertex][adj_list[tempvertex][0]]=non_bound[1];

  /* Reading (just passing on) the information about the
                                      coordinates of vertices */
  for (i = 1; i <= Nvertices; i++)
    fscanf (fin, "%d %d \n", &v1, &v2);

  /* Reading the number of groups of the constrained vertices
                               and then the vertices of each group */
  fscanf (fin, "%d\n", &NConstrained_groups);
  for (i = 1; i <= NConstrained_groups; i++){
    constrained_groups[i][0] = 0;
    while (fscanf (fin, "%d\n", &v1) == 1) {
```

```
        constrained_groups[i][++constrained_groups[i][0]] = v1;
      }
      fscanf (fin, "%s\n", tmpStr);
    }

    /* Check to see if the current configuration is reducible */
    printf("Started!\n");
    if (!Check_Boundary_Colorings (0)) {
      printf ("Configuration No. %d is reducible! No of Colorings
                        Checked = %d\n", current_conf, Nof_colorings);
    }
  }
  fclose (fin);
  fclose (fErrors);
}

/*********************************************************************/
/* If the current configuration is not reducible this procedure writes
   the index of the configuration as well as the coloring of the boundary
   neighbors into a file. */
void UnColorable (void){
  int i;
  printf ("Configuration No. %d is *NOT* reducible \n", current_conf);
  fprintf (fErrors, "Configurtion No %d\n", current_conf);
  fprintf (fErrors, "The coloring of the boundary neighbors that
                              cannot be extended is :\n");
```

```
  for (i = 1; i <= Nbound; i++)
    fprintf (fErrors, "Color of vertex %d = %d\n", bound[i], color[bound[i]]);
  fprintf(fErrors, "\n");
}


/*********************************************************************/
/* Checks whether the current 3-coloring of the boundary neighbors can
   be extended to a 3-coloring of the whole  configuration. Returns 1 when
   it can NOT be extended, 0 otherwise */
int Check_Extendable (int vertex){
  int i, j, Next_vertex, Equal, k;
  char tmp;
  Next_vertex = 0;
  /* Find the "Next vertex" to be colored after coloring the current
     "vertex", by finding an uncolored neighbor of it, if exists any */
  for (i = 1; i <= adj_list[vertex][0]; i++)
    if (!color[adj_list[vertex][i]]) {
      Next_vertex = adj_list[vertex][i];
      i = adj_list[vertex][0];
    }

  /* If all the neighbors of the current "vertex" are colored,  find
                                the next (available) uncolored vertex */
  if (Next_vertex == 0){
    for (i = 1; i <= Nvertices-Nbound; i++)
      if (non_bound[i] != vertex && !color[non_bound[i]]){
        Next_vertex = non_bound[i];
```

```
        i = Nvertices-Nbound;
      }
  }
  /* Check all possible colorings of the current "vertex" and continue
                                         by coloring the "Next_vertex" */
  for (i = 1; i <= 3; i++){
    Equal = 0;
    for (j = 1; j <= adj_list[vertex][0]; j++)
    if (color[adj_list[vertex][j]] == i) Equal = 1;
    if (!Equal){
      color[vertex] = i;
      Ncolored++;
      if (Ncolored == Nvertices || !Check_Extendable (Next_vertex)){
        Ncolored--;
        color[vertex] = 0;
        return 0;
      }
      Ncolored--;
      color[vertex] = 0;
    }
  }
  return 1;
}

/**********************************************************************/
/* Checks whether the current boundary coloring satisfies the
   requirments by the constrained groups. That is, not all the vertices
```

```
   in the same group have the same color. Returns 1 if it does NOT
   satisfy this condition, 0 otherwise. */
int Valid_Boundary_Coloring (void){
  int i, j;
  for (i = 1; i <= NConstrained_groups; i++){
    int All_Equal=1;
    for (j = 2; j <= constrained_groups[i][0]; j++){
      if (color[constrained_groups[i][1]] != color[constrained_groups[i][j]]) {
        All_Equal = 0;
        j = constrained_groups[i][0];
      }
    }
    if (All_Equal) return 1;
  }
  return 0;
}

/********************************************************************/
/* For all possible (valid) colorings of the boundary neighbors checks if
   it is extendable to a coloring of the whole configuration. Returns 1
   if it is NOT, 0 otherwise */
int Check_Boundary_Colorings (int NColored_bound){
  int v1, v2, Equal;

  /* If all boundary neighbors are colored  */
  if (NColored_bound == Nbound) {
    /* check if this coloring of the boundary neighbors satisfies the
```

```
                          requirements by the constrained groups */
  if (NConstrained_groups > 0 && Valid_Boundary_Coloring ()) return 0;
  v1 = non_bound[1];
  v2 = adj_list[non_bound[1]][adj_list[non_bound[1]][0]];

  /* remove one edge, call e, from the configuration */
  adj_list[v1][0]--;
  adj_list[v2][0]--;
  /* first check if the current coloring of boundary neighbors can
                               be extended to a coloring of G-e */
  if (!Check_Extendable (non_bound[1])){
    /* if so then put e back to G and check if this coloring can be
         extended to a coloring of the non-boundary vertices of G */
    adj_list[v1][0]++;
    adj_list[v2][0]++;
    if (!Check_Extendable (non_bound[1])) {
      Nof_colorings++;
      return 0;
    }
    else {
      UnColorable ();
      return 1;
    }
  }
  else {
    /* if the current boundary coloring cannot be extended even to a
                              3-coloring of G-e put e back to G */
```

```
      adj_list[v1][0]++;
      adj_list[v2][0]++;
      return 0;
    }
  }
  else {
    int i, j, MaxColor;
    /* if this is the first boundary neighbor we want to color try
                                                         only color 1 */
    if (NColored_bound == 0) {
      Nof_colorings = 0;
      MaxColor = 1;
    }
    /* if this is the second boundary neighbor we want to color try
                                                  only colors 1 and 2 */
    else if (NColored_bound == 1) MaxColor = 2;
    /* Otherwise, try all possbile 3 colors */
    else MaxColor = 3;
    NColored_bound++;
    for (i = 1; i <= MaxColor; i++){
      Equal = 0;
      for (j=1; j <= adj_list[bound[NColored_bound]][0]; j++)
        if (color[adj_list[bound[NColored_bound]][j]] == i)
          Equal = 1;
      if (!Equal){
        color[bound[NColored_bound]] = i;
        Ncolored++;
```

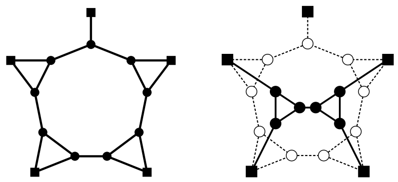
```
        if (Check_Boundary_Colorings (NColored_bound)) {
          color[bound[NColored_bound]] = 0;
          Ncolored--;
          return 1;
        }
        color[bound[NColored_bound]] = 0;
        Ncolored--;
      }
    }
    return 0;
  }
}
/*********************************************************************/
int main (int argc, char *argv[]){
  time_t start_time = time(NULL);
  if (argc >= 2)
    Read_Data (argv[1]);
  else Read_Data ("conf.dat");
  printf ("All done in %g seconds!\n", difftime(time(NULL), start_time));
  return 0;
}
```

Appendix C

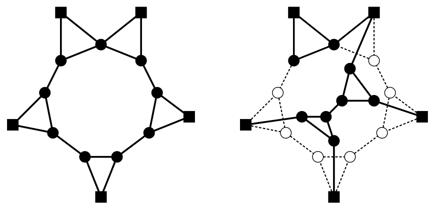
List of Reducible Configurations for Theorem 3.1.1

The first three reducible configurations in the proof of Theorem 3.1.1 are the ones that were also used in the proof of Theorem 3.2.1; a ≤ 2-vertex, a cut-vertex, and a $2k$-face with at least $2k-1$ bad vertices. Here is the list of the other 74 reducible configurations, including all subconfigurations of the configurations listed in Section 3.3.2. We have listed them in twelve groups, each corresponding to a configuration listed in Section 3.3.2. For each group that contains at least two subconfigurations, we explain how the list is generated. Each graph that has white vertices and dotted edges is the "*modified*" version (by removing some vertices and edges and adding a gadget) of the graph to its left. The vertices and the edges that have been removed are the white vertices and the dotted edges, respectively.

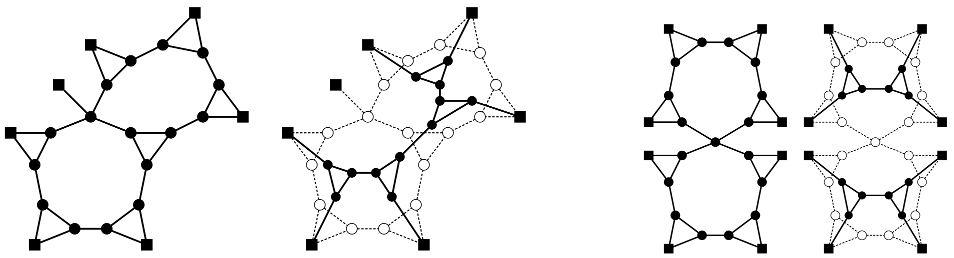
1- Simple face: There is only one possible case:



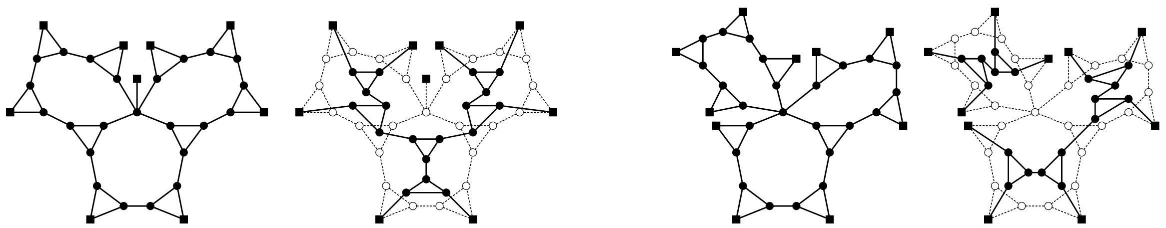
2- Type 2 face: There is only one possible case:



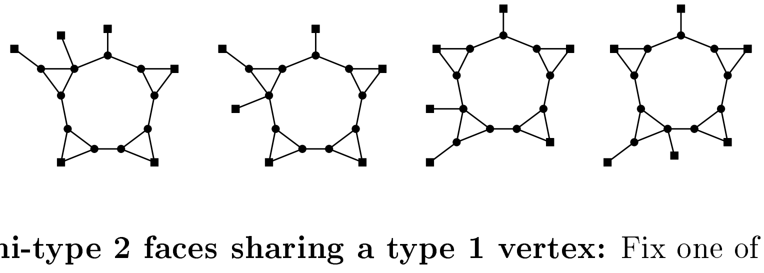
3- Two type 0 faces sharing their type 0 vertex: It is easy to see that there are two possible configurations:



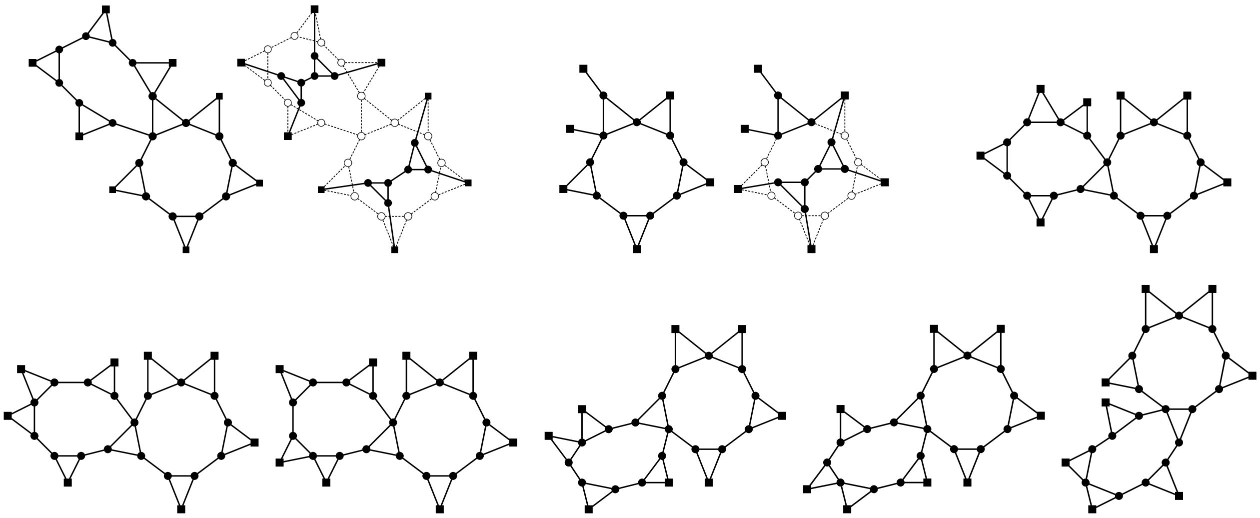
4- Three type 5 faces sharing a 5-vertex: There are only two possible configurations of this type:



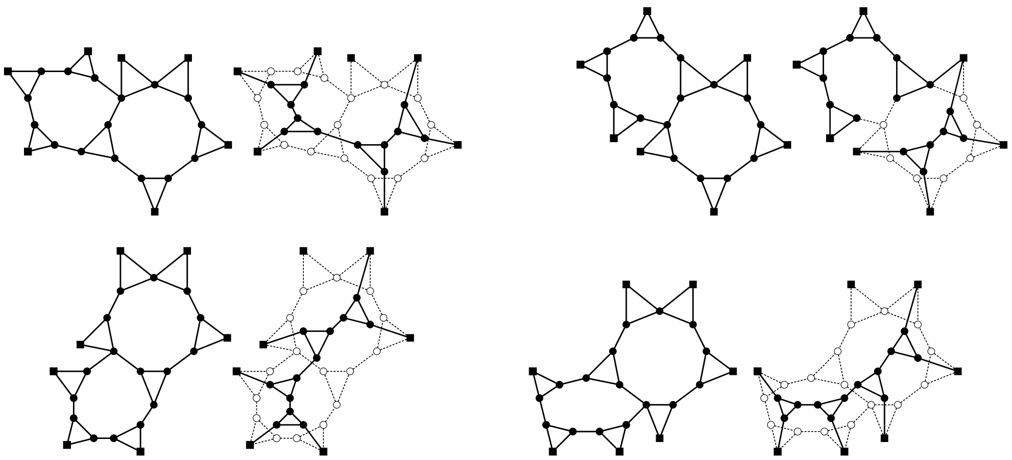
5- Two semi-simple faces sharing a type 1 vertex: Instead, we consider the following configurations. It is easy to see that if we fix one of the semi-simple faces, based on the location of its type 1 vertex we obtain one of the following structures:



6- Two semi-type 2 faces sharing a type 1 vertex: Fix one of the semi-type 2 faces, and consider different locations for its type 1 vertex, moving it around the boundary of the face in counter-clockwise order. For each such case, by moving the position of the type 2 vertex in the other face (in counter-clockwise order) we obtain the following eight configurations.

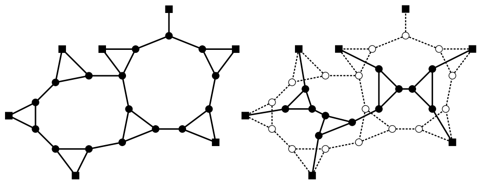


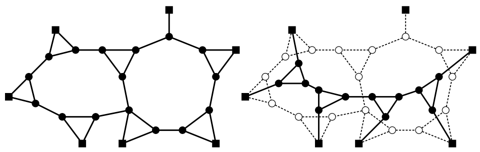
7- A semi-type 2 face sharing its type 1 vertex with a type 1 face: Again, fix the semi-type 2 face, and consider different positions of its type 1 vertex, moving it around the face in counter-clockwise order. There are four possible configurations of this kind.

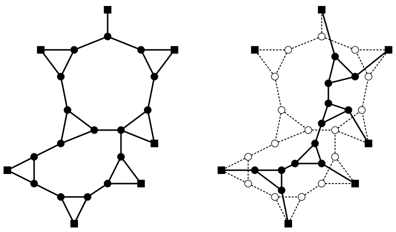


8- A semi-type 2 face sharing its type 1 vertex with a semi-simple face: It is straightforward to check that there are five possible configurations of this kind. Four of them are the same as the ones in item 5 above, and the other contains the second configuration of item 6 above.

9- A semi-simple face sharing its type 1 vertex with a type 1 face: Fix the semi-simple face and consider different locations of its type 1 vertex, moving it around the face in counter-clockwise order. There are three configurations of this kind.

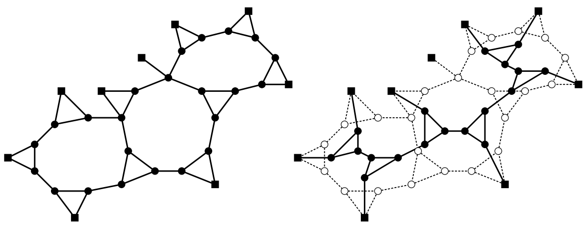


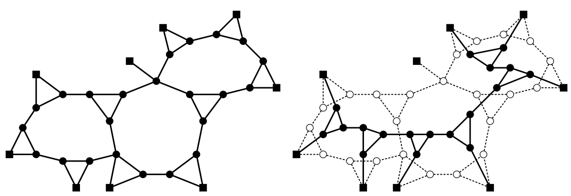


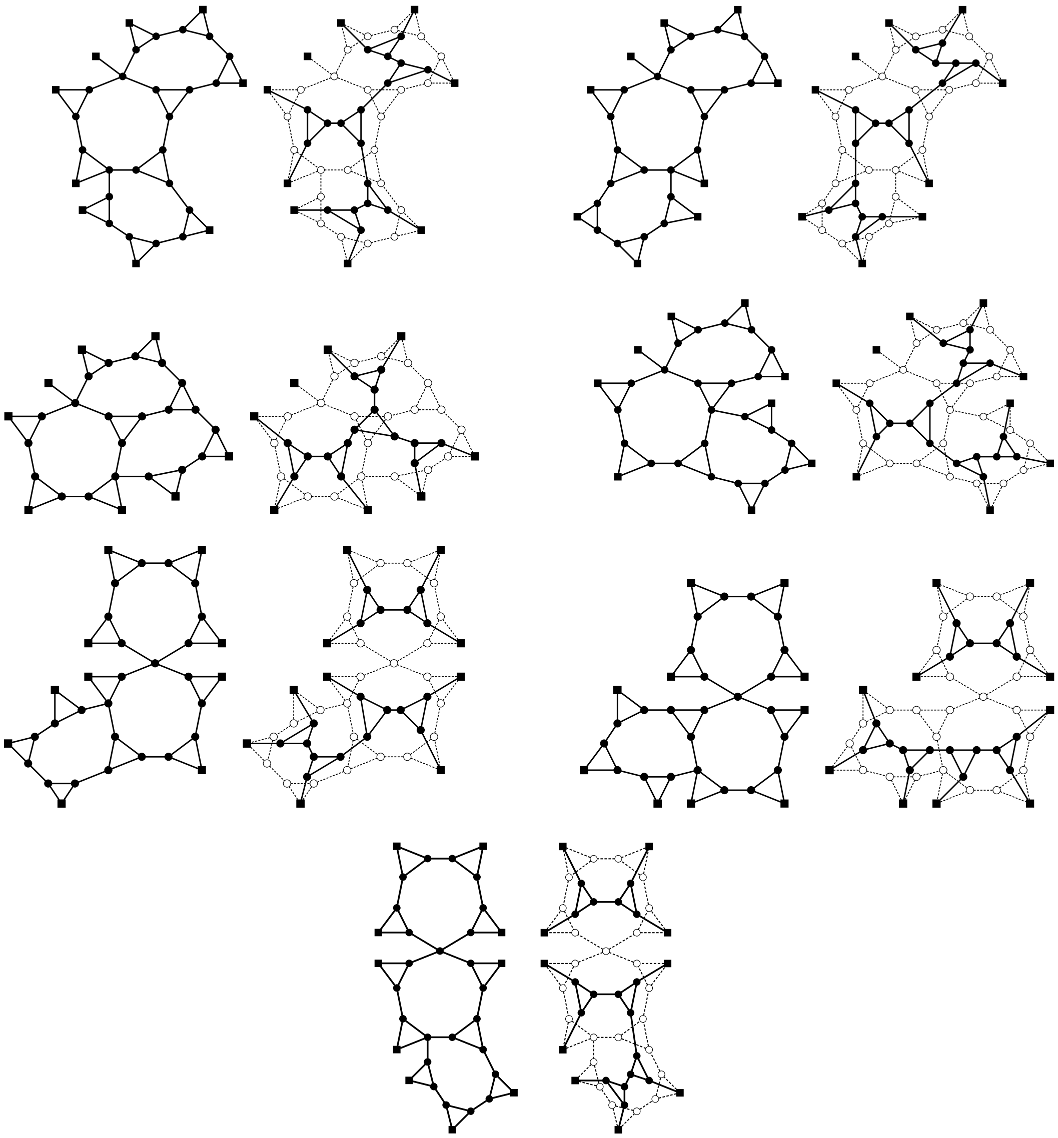


10- Simple triple structure: It is easy to see that the semi-simple face of a simple triple structure has one of the four possible structures given in item 5 above. Thus, the reducibility of any simple triple structure follows from part 8 of Lemma 3.3.7 and we don't need to consider different possibilities for a simple triple structure.

11- Triple structure of kind 1: There are nine configurations of this kind. First assume that the semi-type 0 and the type 0 face are sharing an edge. Then based on the location of the type 1 vertex of the semi-type 0 face and moving it around the face in counter-clockwise order, we obtain the first six configurations listed below. In the next three configurations the semi-type 0 face and the type 0 face do not share any edges. It is easy to see that there are only three configurations of this kind (listed below) up to isomorphism.







12- Triple structure of kind 2: First assume that the semi-type 0 and the type 0 face are sharing an edge. We consider all possible locations for the type 1 vertex of the semi-type 0 face, moving it around the face in counter-clockwise order. The first configuration below is when the type 1 vertex is adjacent to the type 0 vertex. If the type 2 vertex of the semi-type 2 face is any vertex other than the one in the figure, then the configuration will contain the second configuration we gave for group 6.

The rest of the configurations are obtained by considering all possible locations for the type 2 vertex of the semi-type 2 face (again moving it around in counter-clockwise order). We do a similar thing for the case that the semi-type 0 face and the type 0 face are not sharing an edge.