CMPUT 671 Assignment 2
Due Oct. 23, 2003

1. Let $V = \{1, \ldots, n\}$ and assume the $\mathcal{G}_{np}$ distribution. Show that for fixed $\epsilon > 0$ if $p \in O\left(\frac{1}{n^{2/3+\epsilon}}\right)$ then the probability of a 4-clique occurring in a random graph goes to zero as n goes to infinity.

   Hint: use the first moment method; that is show the expected number of 4-cliques goes to 0.

2. Using $\mathcal{G}_{np}$ as in the previous question, show that there is a constant c such that if $p \geq c/n$ then the probability that G is 3-colorable goes to zero as n goes to infinity.
   Hint: use the first moment method to show the probability of an independent set of sufficient size goes to zero. (i.e. there is a lower bound on the smallest possible largest set in a 3-colorable graph).

3. Note that the previous two questions say there are large sparse (e.g. $p \in O(1/n)$) random graphs that are not 3-colorable but at the same time have no 4-clique, which is one thing we might look for to prove non-colorability. A 4-clique is the smallest example of a 4-critical graph; that is, it is not 3-colorable, but if any edge is deleted then it is 3-colorable. The wheels with odd cycles used in the first assignment are all 4-critical as well. (The 3-critical graphs are just the odd cycles, and so easy to detect.)

   In the following we will determine a lower bound on the size of the smallest non-3-colorable graph (with high probability) when $p = c/n$.

   (a) Show that a critical graph must have minimum degree of at least 3.

   (b) Suppose you are looking for a critical graph on s vertices. Use (a) to give a lower bound on the number of edges in any such graph.

   (c) For edge probability p, provide a formula to upper bound the probability $p(s)$ that an arbitrary set of s vertices contains a critical subgraph, by bounding the probability of it having sufficient edges.

   (d) For a graph $G = (V, E)$, a subset $S \subseteq V$ induces the subgraph $G[S] = (S, E')$ where $E' = \{(x, y) | x, y \in S, (x, y) \in E\}$.
   Given a graph G with n vertices generated randomly with $p = c/n$, give a formula to upper bound the expected number, called $a(n, s)$, of subgraphs induced by a set of s vertices that contain a critical graph. For this part assume s is a fixed constant.

   (e) At this point you should have a big messy formula for $a(n, s)$. Show that the sum of the expected values $a(n, s)$ for $4 \leq s \leq h$ goes to zero as n goes to infinity for suitable functions $h = h(n)$. It should not be too hard to show there exists an $\epsilon > 0$ such that this is true for $h < n^{\epsilon}$.

   (If you are familiar with more advanced combinatorial methods, you can do better, i.e. $h = \alpha n$ for a suitable small $\alpha > 0$. For example, you may split the sum as is done in in lemma 1 of the Chvátal and Szemerédi, *Many Hard Examples for Resolution* JACM Vol. 35(4) 1988, pp 759–768.)

You have shown that the smallest critical graph is large, which suggests that 3-coloring will be hard at the threshold.

4. The Hajós construction: Let K_k be the complete graph on k vertices. Assume we start our collection with K_k. Repeatedly select one of the following steps to create new G to add to the collection. (To identify x, y, we replace them by a single vertex z, and put $(z, w) \in E$ if $(x, w) \in E$ or $(y, w) \in E$, $w \neq x$ or y.)

   (a) If G_1 and G_2 are in the collection (they may copies of the same graph) remove an edge (x_1, y_1) from G_1 and an edge (x_2, y_2) from G_2. Let G be the union, identify vertices x_1 and x_2 and add the edge (y_1, y_2).

   (b) If G_1 is in the collection and it contains a non-adjacent pair of vertices x, y, then construct G by identifying x and y.

   (c) If G_1 is in the collection then create G by adding new vertices and/or new edges to G_1.

   Consider the following questions.

   (a) Prove that every graph constructible using this method has chromatic number at least k.

   (b) Let $k = 4$. Show a sequence of construction steps to generate a non-3-colorable graph which is triangle free.

   (c) Let $k = 4$ and restrict the construction to type (a) steps only. Prove a linear upper bound on the number of edges as a function of n as n becomes large.

   (d) Let $k = 4$ and restrict the construction to type (a) steps only. Show that every graph constructed is 4-critical.

   (e) Let $k = 4$ and restrict the construction to type (a) steps only. Show that there exists a polynomial sized proof (not necessarily a resolution proof) that the graph is not 3-colorable.

   (f) * Let $k = 4$ and restrict the construction to type (a) steps only to construct a graph H. Let $G \supseteq H$. Give a polynomial time algorithm to show any such G is not 3-colorable.

   This exercise shows that large sparse critical subgraphs are not necessarily hard to identify. Hajós showed in 1961 that every graph with chromatic number at least k could constructed by his method. It is open as to whether or not there is always a polynomial (in n) Hajós construction for any non-k-colorable graph, but a positive answer would imply NP = CO-NP, so is considered unlikely. Note that steps of type (b) reduce the number of vertices, which is "why" the complete construction may not be polynomial in n.