Succinct Games and Exotic Numeration Systems

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Extended Abstract

1 Introduction

For simplicity, we restrict attention to 2-player, 0-sum perfect information games without chance moves which are *acyclic*. Further, we consider normal play, i.e., the player making the last move wins, and the opponent loses. The tool for providing a polynomial strategy for this class of games is the *Sprague-Grundy* function [2], g-function for short: for a general digraph with n edges, the g-function can be computed in $O(n)$ steps.

A game is *succinct* if its input size is $O(\log n)$ rather than $O(n)$. An example of a succinct game is *Nim*, in which a finite number n of *tokens* (marbles, stones or the like) is arranged in $k \geq 1$ heaps (piles). A move consists of selecting a heap and removing from it a positive number of tokens, possibly the entire heap. The input size is $\sum_{i=1}^{k} \log n_i$, where $n_1, \ldots, n_k$ are the sizes of the k heaps. Because of succinctness, an additional property of the g-function is required to reduce the complexity from $O(\sum_{i=1}^{k} n_i)$ to $O(\sum_{i=1}^{k} \log n_i)$. The fact that the g-values of Nim are arranged in a simple arithmetic sequence constitutes this additional property. For the polynomial subclass of the class of *octal* games [9], polynomiality is usually established by showing that the g-function is periodic, though the period and/or preperiod can sometimes be very large [8].

In this note we show that for certain succinct games for which the g-function is highly chaotic, polynomiality can nevertheless be established by resorting to special numeration systems. We close with a question.

In any game, an N-position is any position from which the *N*ext (first) player can win, independent of the moves of the opponent. A P-position is any position such that the *P*revious (second) player can win, independent of the moves of the opponent. The set of all N-positions of a game is denoted by $\mathcal{N}$, and the set of all P-positions by $\mathcal{P}$.

2 An Example

Denote by $\mathbb{Z}^0$ the set of nonnegative integers, and by $\mathbb{Z}^+$ the set of positive integers. Define a family of succinct games, played on two heaps of tokens, which depends on two parameters $s, t \in \mathbb{Z}^+$. There are two types of moves: I. Take any positive number of tokens from a single heap, possibly the entire heap. II. Take $k > 0$ and $l > 0$ from the two heaps, where, say, $0 < k \leq l$. This move is constrained by the condition $0 < k \leq l < sk + t$, which is equivalent to $0 \leq l - k < (s-1)k + t, \quad k \in \mathbb{Z}^+$. The case $s = t = 1$ is known as *Wythoff's game* [10], [4], [11].

For $s = t = 2$, the first few P-positions are listed in Table 1. What's its next entry?

Table 1: The first few P-positions for $s = t = 2$.

n	A_n	B_n
0	0	0
1	1	4
2	2	8
3	3	12
4	5	18
5	6	22
6	7	26
7	9	32
8	10	36
9	11	40
10	13	46
11	14	50
12	15	54
13	16	58

Let S be any finite subset of $\mathbb{Z}^0$. Define $\text{mex}\, S = \mathbb{Z}^0 \setminus S =$ least nonnegative integer not in S. In [7], where the game has been proposed and analyzed, the following was proved:

Theorem 1. $\mathcal{P} = \bigcup_{i=0}^{\infty}\{(A_i, B_i)\}$, *where* $A_n = \text{mex}\{A_i, B_i : 0 \leq i < n\}$ *and* $B_n = sA_n + tn$ $(n \geq 0)$.

It is easy to see that if $A = \bigcup_{n=1}^{\infty} A_n$, $B = \bigcup_{n=1}^{\infty} B_n$, then A and B are *complementary*, i.e., $A \cup B =$ set of all positive integers, and $A \cap B = \emptyset$.

Given any two heaps of the game, containing x and y tokens with $x \leq y$. The complementarity of A and B implies that either $x = A_n$ or $x = B_n$ for some n. Hence Table 1 has to be computed only up to the encounter of x. Moreover, it is not hard to see that $n \leq x$, and if $x = A_n$, then $x/2 < n$, so the table has to be computed up to at most $\Omega(x)$, which implies a strategy computation linear in x. But because of succinctness, this is an exponential strategy! We now show that a linear strategy of complexity $O(\log x)$ exists, which is based on a special numeration system.

For fixed $s, t \in \mathbb{Z}^+$, put $u_{-1} = 1/s$, $u_0 = 1$, and let $u_n = (s+t-1)u_{n-1} + su_{n-2}$ $(n \geq 1)$. Denote by $\mathcal{U}$ the numeration system with bases $u_0, u_1, \ldots$ and digits $d_i \in \{0, \ldots, s+t-1\}$, with the additional requirement $d_{i+1} = s+t-1 \Rightarrow d_i < s$ $(i \geq 0)$. Every positive integer has a unique representation over $\mathcal{U}$, [5].

Consider the special case $s = t = 2$. Then $u_{-1} = \frac{1}{2}$, $u_0 = 1$, $u_1 = 4$, $u_2 = 14$, $u_3 = 50$, $u_4 = 178$, $\ldots$. The representations of the integers 1 to 60 in this numeration system are displayed in Table 2.

Table 2: A quaternary representation of the first few integers in $\mathbb{Z}^+$.

50	14	4	1	n	14	4	1	n
	2	0	3	31			1	1
	2	1	0	32			2	2
	2	1	1	33			3	3
	2	1	2	34		1	0	4
	2	1	3	35		1	1	5
	2	2	0	36		1	2	6
	2	2	1	37		1	3	7
	2	2	2	38		2	0	8
	2	2	3	39		2	1	9
	2	3	0	40		2	2	10
	2	3	1	41		2	3	11
	3	0	0	42		3	0	12
	3	0	1	43		3	1	13
	3	0	2	44	1	0	0	14
	3	0	3	45	1	0	1	15
	3	1	0	46	1	0	2	16
	3	1	1	47	1	0	3	17
	3	1	2	48	1	1	0	18
	3	1	3	49	1	1	1	19
1	0	0	0	50	1	1	2	20
1	0	0	1	51	1	1	3	21
1	0	0	2	52	1	2	0	22
1	0	0	3	53	1	2	1	23
1	0	1	0	54	1	2	2	24
1	0	1	1	55	1	2	3	25
1	0	1	2	56	1	3	0	26
1	0	1	3	57	1	3	1	27
1	0	2	0	58	2	0	0	28
1	0	2	1	59	2	0	1	29
1	0	2	2	60	2	0	2	30

A question we just might ask at this point is whether there is any connection between Tables 1 and 2. If we scan the first few entries of both, we may be tempted to conclude that all the entries under A_n in Table 1 have representations ending in no 0 in Table 2. But then 14 is a counterexample, whose representation ends in two 0s. Also it appears that the B_n all have representation ending in a single 0. But 50, with representation 1000 is a counterexample, in fact, the only counterexample in the range of the two tables.

It turns out, however, that the following two remarkable, æsthetically pleasing, properties hold in general:

a. All the A_n have representations ending in an *even* number of 0s, and all the B_n have representations ending in an *odd* number of 0s.

b. For every $(A_n, B_n) \in \mathcal{P}$, the representation of B_n is the "left shift" of the representation of A_n.

Thus $(1,4)$ of Table 1 has representation $(1,10)$, and $(6,22)$ has representation $(12,120)$: 10 is the "left shift" of 1, 120 the left shift of 12. We remark that the second part of **a** follows from its first part, since A and B are complementary.

We leave it to the reader to show that these observations lead to an easy polynomial strategy for our class of games. We remark that for Wythoff's game ($s = t = 1$), and even for a generalization thereof [6] ($s = 1,\ t \geq 1$), there is another polynomial strategy, based on the fact that then $A_n = \lfloor n\alpha \rfloor$, $B_n = \lfloor n\beta \rfloor$, where $\alpha = (2 - t + \sqrt{t^2+4})/2,\ \ \beta = (2 + t + \sqrt{t^2+4})/2$. But for $s > 1$, there exist no such α, β, and we don't know whether there is a polynomial strategy other than the one based on the above numeration system.

3 Another Numeration System

A special case of a numeration system considered in §4 of [5] is based on the even-indexed Fibonacci numbers, namely, $1, 3, 8, 21, 55, 144, \ldots$. They satisfy the recurrence $p_n = 3p_{n-1} - p_{n-2}$ ($n \geq 1$), where $p_{-1} = 0$, $p_0 = 1$. Every positive integer N has a unique representation in a ternary numeration system of the form $N = \sum_{i \geq 0} d_i p_i$, where the digits satisfy $d_i \in \{0,1,2\}$ ($i \geq 0$), and the additional condition: if for some $0 \leq k < l \leq n$, $d_l = d_k = 2$, then there is j with $k < j < l$ such that $d_j = 0$. The representations of the first few positive integers are given in Table 3. This numeration system has also been used in [3].

In Table 4 we display the first few positive integers A_n whose representation in this ternary numeration system ends in an even number of 0s, and the numbers B_n whose representation ends in an odd number of 0s.

The following property seems to hold: $A_n = \text{mex}\{A_i, B_i : 0 \leq i < n\}$, $B_n = 2A_n + n - r_n$, where $r_n = |\{k : B_k - B_{k-1} = 2,\ k \leq n\}|$. Is there a game with two heaps and *simple* rules such that $\mathcal{P} = \bigcup_{i=0}^{\infty}\{(A_i, B_i)\}$? That is, the rules should be independent of the *size* of either heap.

Remarks.

- The special case $s = t = 2$ of the (s,t)-sequences considered above was recently put into Neil Sloane's *On-Line Encyclopædia of Integer Sequences*

Table 3: A ternary representation of the first few integers in $\mathbb{Z}^+$.

55	21	8	3	1	n	21	8	3	1	n
	1	1	0	0	29				1	1
	1	1	0	1	30				2	2
	1	1	0	2	31			1	0	3
	1	1	1	0	32			1	1	4
	1	1	1	1	33			1	2	5
	1	1	1	2	34			2	0	6
	1	1	2	0	35			2	1	7
	1	1	2	1	36		1	0	0	8
	1	2	0	0	37		1	0	1	9
	1	2	0	1	38		1	0	2	10
	1	2	0	2	39		1	1	0	11
	1	2	1	0	40		1	1	1	12
	1	2	1	1	41		1	1	2	13
	2	0	0	0	42		1	2	0	14
	2	0	0	1	43		1	2	1	15
	2	0	0	2	44		2	0	0	16
	2	0	1	0	45		2	0	1	17
	2	0	1	1	46		2	0	2	18
	2	0	1	2	47		2	1	0	19
	2	0	2	0	48		2	1	1	20
	2	0	2	1	49	1	0	0	0	21
	2	1	0	0	50	1	0	0	1	22
	2	1	0	1	51	1	0	0	2	23
	2	1	0	2	52	1	0	1	0	24
	2	1	1	0	53	1	0	1	1	25
	2	1	1	1	54	1	0	1	2	26
1	0	0	0	0	55	1	0	2	0	27
1	0	0	0	1	56	1	0	2	1	28

Table 4: The first few P-positions — of which game?

n	A_n	B_n
1	1	3
2	2	6
3	4	11
4	5	14
5	7	19
6	8	21
7	9	24
8	10	27
9	12	32
10	13	35
11	15	40
12	16	42
13	17	45
14	18	48

at http://www.research.att.com/~njas/sequences/index.html , sequence numbers A045671, A045672.

- A sequence defined quite differently, namely A026366, turns out to be equivalent to an (s,t)-sequence with $s=2$, $t=1$. In [7] this definition was generalized to any $s,t \in \mathbb{Z}^+$, and the equivalence to the above (s,t)-sequences was proved.

- In [1], an interpretation for the recurrence $f_{n+1} = 6f_n - f_{n-1}$ was found. Other interpretations, in terms of numeration systems belonging to the same family as the system considered last, can be given. We give a few examples. Consider this recurrence with $f_0 = 1$ throughout. If we put $f_1 = 7$, we get a numeration system with digits satisfying $0 \le d_i \le 5$ $(i \ge 1)$, $0 \le d_0 \le 6$ and: if d_k and d_l are maximal, then there is j with $k < j < l$ such that $d_j < 4$. If $f_1 = 6$, the numeration system digits satisfy $0 \le d_i \le 5$ $(i \ge 0)$and the same additional condition. If $f_1 = 9$, then $0 \le d_i \le 5$ $(i \ge 1)$, $0 \le d_0 \le 8$ and the same additional condition.

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