CMPUT 675: Approximation Algorithms **Winter 2018**

Lecture 10 (Feb. 8, 2018): Facility Location via Primal/Dual

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10.1 Uncapacitated Facility Location

Recall that in the metric uncapacitated facility location problem (UFL) we have:

- Input
  - F : a set of facilities
  - C : a set of clients
  - c_{ij} : the cost of serving client j at facility i which is a metric
  - f_i : the opening cost of facility i
- Goal: select a subset of facilities to open and an assignment of clients to open facilities so as to minimize the total cost of the open facilities and the assignments costs.

We learned a randomized approximation algorithm for the UFL problem in Lecture 7. In this lecture, we will study a primal-dual approximation algorithm for this problem. The following are a natural LP relaxation for UFL and its corresponding dual.

$$\begin{aligned}
\text{minimize} \sum_{i \in F} f_i y_i \quad &+ \quad \sum_{i \in F, j \in C} c_{ij} x_{ij} && (10.1)\\
\text{s.t.} \sum_{i \in F} x_{ij} \quad &\geq \quad 1 \qquad \forall j \in C && (10.2)\\
y_i \quad &\geq \quad x_{ij} \qquad \forall i \in F, j \in C && (10.3)\\
x_{ij} \quad &\geq \quad 0 \qquad \forall i \in F, j \in C && (10.4)\\
y_i \quad &\geq \quad 0 \qquad \forall i \in F && (10.5)
\end{aligned}$$

wherevariable x_{ij} indicates whether client j is assigned to facility i, and the variable y_i indicates whether open facility i or not. The dual LP is:

$$\begin{aligned}
\text{maximize} \sum_{j \in C} \alpha_j \quad & && (10.6)\\
\text{s.t.} \sum_{j \in C} \beta_{ij} \quad &\leq \quad f_i \qquad \forall i \in F && (10.7)\\
\alpha_j - \beta_{ij} \quad &\leq \quad c_j \qquad \forall i \in F, j \in C && (10.8)\\
\beta_{ij} \quad &\geq \quad 0 \qquad \forall i \in F, j \in C && (10.9)\\
\alpha_j \quad &\geq \quad 0 \qquad \forall j \in C && (10.10)
\end{aligned}$$

Intuitively, we can consider the variable α_j as the total amount that client j will pay and this will be used to pay for the serivce cost of client j as well as how much that client pays towards openning the facility it is getting service at. Since each facility has an opening cost, we can split the cost f_i into non-negative shares β_{ij} apportioned among clients who get served at this facility. Each client j would like to pay the lowest service cost among all facilities and its share of the facility opening cost, i.e. $\alpha_j = min_{i \in F}(c_{ij} + \beta_{ij})$. The complimentary slackness conditions for the primal and dual LP imply:

(i) $\forall i, j : x_{ij} > 0 \to \alpha_j - \beta_{ij} = c_{ij}$

(ii) $\forall i : y_i > 0 \to \sum_{j \in C} \beta_{ij} = f_i$

(iii) $\forall j : \alpha_j > 0 \to \sum_{i \in F} x_{ij} = 1$

(iv) $\forall i, j : \beta_{ij} > 0 \to y_i = x_{ij}$

10.2 Algorithm

The algorithm starts with the trivial (infeasible) primal solution of all values zero and the trivial dual solution of all zero. It tries to (iteratively) improve feasibility of the primal and optimality of the dual. It will maintain an integer primal solution but a fractional feasible dual solution. At the end we show that we have a primal integral and a dual solution (both feasible) whose costs are close to each other. Hence a good approximation algorithm for UFL. The algorithm consists of two phases. In phase 1, the algorithm tries to find a set of tight edges and temporarily open facilities, F_t. In phase 2, some facilities get pruned and a set $I \in F_t$ is selected to permanently open, a mapping ϕ from Clients to I is found. To better demonstrate the algorithm, some more definitions are necessary:

Definition 1 *An edge ij for $i \in F$ and $j \in C$ is said to be* **tight** *if the constraint (10.8) holds with equality. A facility $i \in F$ is* **paid for** *if $\sum_{j \in C} \beta_{ij} = f_i$. A facility i is said to be the* **witness** *of a client j if j gets connected to i in phase 1.*

Phase 1:

Start from all zero Primal-Dual solution. Raise α_j values uniformly for all "unserviced" clients until $\alpha_j = c_{ij}$ for some i, j. (edge goes tight, i.e constraint (10.8) holds with equality). At this point, we start raising β_{ij} for the tight edge at the same time with the same raite as α_j is raised to maintain constraint (10.8). The next type of event that can happen is if constraint (10.7) goes tight for some facility $i \in F$, i.e. i is "paid for". If facility $i \in F$ is paid for, we temporarily open i and all "uncovered/unserviced" $j \in C$ with a tight edge are connected to i. (i is witness for j). We freeze all these clients dual values. Whenever a client j' has tight edge to i, it goes connected to i. (α'_j pays for $c_{ij'}$ when i is already paid for) The algorithm terminates when all clients are connected.

Note: As shown in Figure 10.1, a client might pay towards opening multiple facilities, but gets connected to only one of them. However, we want to ensure that each client only pay for the facility that it is eventually connected to. This can be ensured in Phase 2 by selecting a subset of the temporarily facilities to open permanently.

Phase 2:

Let F_t be the set of temporarily open facilities and T be the set of tight edges in G. We build T^2 s.t $(i, i') \in T^2$ iff i and i' have a common neighbour in T. So two facilitys $i, i' \in T$ will be neighbors in T^2 if there is a client j that had tight edges to both i, i'. Let H be the subgraph of G induced on F_t with edge set T^2. We find a maximal independent set I in H and open these facilities permanently. For each client $j \in C$:

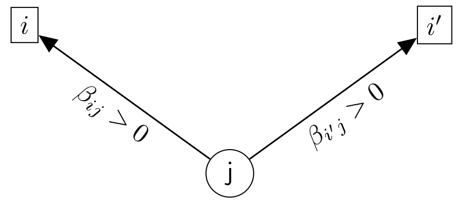


Figure 10.1: Client j might have $\beta_{ij} > 0$ and $\beta_{i'j} > 0$

- if $\beta_{ij} > 0$ (i.e ij was tight) and $i \in I$, then $\phi(j) = i$ (we say j is directly connected).
- Otherwise, consider the tight edge (i', j) s.t i' was witness for j.
  - if $i' \in I$, then set$\phi(j) = i'$ (this is again a directly connected client but we might have had $\beta_{i'j} = 0$)
  - if $i' \notin I$, let i be any neighbour of i' in graph H s.t. $i \in I$, and set $\phi(j) = i$. (indirectly connected)

Note: I and ϕ form a primal integral solution: we set $x_{ij} = 1$ iff $\phi(j) = i$, and $y_i = 1$ iff $i \in I$. And the values of α_j and β_{ij} obtained from Phase 1 form a dual feasible solution.

10.3 Analysis of the algorithm

We will show how the dual variable α_j's pay for the primal cost of opening facilities and connecting clients. Let us break down α_j into two parts: $\alpha_j = \alpha_j^f + \alpha_j^e$, where α_j^f is the portion of opening cost that the client j has paid, and α_j^e is the portion of assignment cost. If j is indirectly connected, then $\alpha_j^f = 0$, $\alpha_j = \alpha_j^e$; and if j is directly connected, then $\alpha_j^f = \beta_{ij}$, $\alpha_j^e = c_{ij}$.

Lemma 1 $\forall\ i \in I$: $\sum_{j:\phi(j)=i} \alpha_j^f = f_i$

Proof. Since i was temporarily open at the end of Phase 1, it is paid for by clients that have tight edge to it in Phase 1. That is:

$$\sum_{j:\beta_{ij}>0} \beta_{ij} = f_i.$$

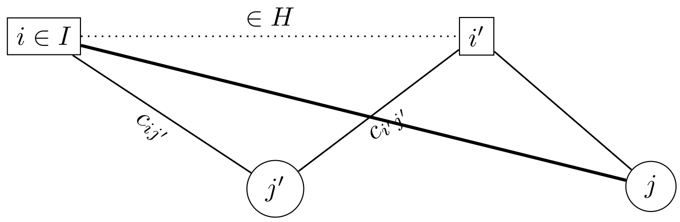
All such j's are connected to i directly: $\alpha_j^f = \beta_{ij}$. For j's connected indirectly, $\alpha_j^f = 0$. The lemma follows. ■

Corollary 1 *Since only the directly connected clients pay for the cost of opening facilities, we can conclude that*

$$\sum_{j \in C} \alpha_j^f = \sum_{i \in I} f_i.$$

Lemma 2 *For any client j that is connected indirectly, $c_{ij} \leq 3\alpha_j^e$ where $\phi(j) = i$.*

Proof. Suppose i' was the witness for client j. Since j is indirectly connected to i, (i, i') must be an edge in H. Therefore, $\exists j'$ such that both $\beta_{ij'} > 0$ and $\beta_{i'j'} > 0$. Since edges (i', j') and (i, j) are tight, $\alpha_{j'} \geq c_{ij'}$ and $\alpha_{j'} \geq c_{i'j'}$. Suppose i and i' were temporarily open in Phase 1 at times t and t', respectively. During Phase 1, $\alpha_{j'}$ stops growing as soon as one of facilities that j' has a tight edge to opens. Therefore, $\alpha_{j'} \leq \min\{t, t'\}$. Also, since i' is the witness for j, we raised α_j until i' became open, $\alpha \geq t'$. Therefore, we can conclude $\alpha_j \geq \alpha_{j'}$. Thus $\alpha_j \geq c_{ij'}$ and $\alpha_j \geq c_{i'j'}$, then, by the triangle inequality, we can conclude that $3\alpha_j^e \geq c_{ij}$, and the required inequality holds. ■



Theorem 1 *The primal-dual algorithm is a 3-approximation algorithm for the uncapacitated facility location problem.*

Proof. For directly connected client j: $c_{ij} \leq \alpha_j^e$ as this holds by constraints (10.8). For indirectly connected client j: $c_{ij} \leq 3\alpha_j^e$ (by Lemma 2). Also, using Corollary 1: $3\sum_{i \in I} f_i = 3\sum_{j \in C} \alpha_j^f$. Putting everything together, we have the cost of the solution is at most:

$$3\sum_{i \in I} f_i + \sum_{i \in I, j \in C} c_{ij} \leq \sum_{j \in C}(3\alpha_j^f + 3\alpha_j^e) \leq 3\sum_{j \in C} \alpha_j \leq 3\mathrm{OPT}$$

Thus, we conclude that the primal-dual algorithm is a 3-approximation algorithm for the UFL problem. ∎