Lecture 27: Graph Algorithms

Agenda:

- Greedy algorithms: elements & properties
- Minimum spanning tree
- 1^{st} algorithm — Prim's

Reading:

- Textbook pages 379 – 384, 558 – 579

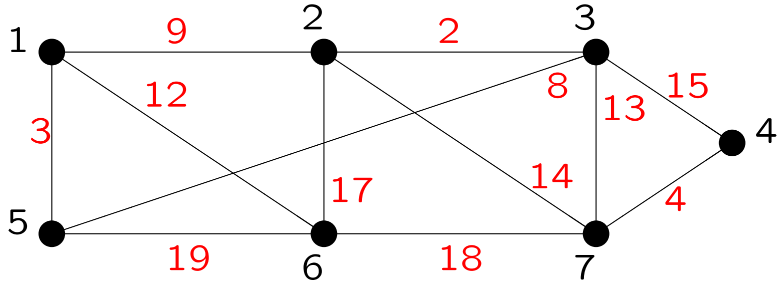
Minimum spanning tree (MST) problem:

- Input: edge-weighted (simple, undirected) connected graphs (positive weights)
- Notions:
  - subgraph, acyclic, tree
  - spanning subgraph: subgraph including all the vertices
  - spanning tree: spanning subgraph which is a tree — acyclic connected subgraph $T = (V, E')$, where $E' \subset E$

    e.g., BFS/DFS (on a connected input graph) tree is a spanning tree of the graph
  - minimum spanning tree: minimum weight
- The MST Problem:

  Find a minimum spanning tree for the input graph.

  For example:



- The minimum spanning forest problem:

  The given graph is not necessarily connected.

  Find an MST for each connected component.

Greedy algorithms and MST problem:

- Greedy algorithms:
  - greedy — each step makes the best choice (locally maximum)
  - iterative algorithms
  - optimal substructure

    an optimal solution to the original problem contains within it optimal solutions to subproblems

- Greedy solution may NOT be globally optimum

  e.g., matrix-chain multiplication: $A_{6\times 5} \times A_{5\times 2} \times A_{2\times 5} \times A_{5\times 6}$

  Greedy: $50 + 150 + 180 = 380$ scalar multiplications

  Dynamic programming: $60 + 60 + 72 = 192$ scalar multiplications

- The MST problem:

  Two greedy solutions are globally optimum

  - Prim's (Prim + Dijkstra + Boruvka's)

    growing the tree to include more vertices
  - Kruskal's (Kruskal + Boruvka's)

    growing the forest to become a tree

Prim's algorithm for the MST problem:

- Input: an edge-weighted (simple, undirected, connected) graph (positive weights)
- Output: an MST
- Idea:
  - suppose we have already an MST T' spanning subset V' of vertices (T' is initialized empty and V' is initialized to contain any one vertex)
  - grow T' to span one more vertex $v \in V - V'$
  - v is selected such that there is a vertex $u \in V'$, edge (u, v) is the minimum weighted over all edges of form (u', v') where $u' \in V'$ and $v' \in V - V'$
  - when V' becomes V, terminate
- One simplest implementation:

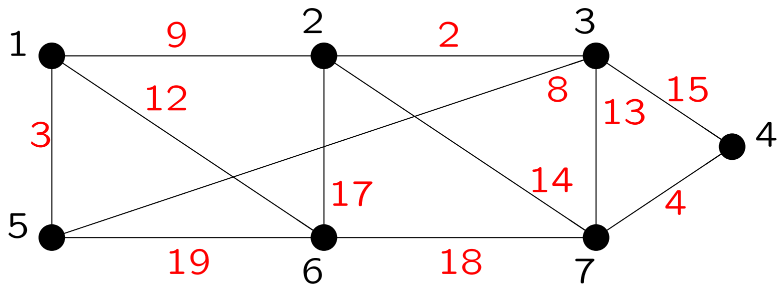
```
procedure primMST(G, w, r)          **G = (V, E)

S = {r}                             **S spanned vertex subset
T = ∅                               **T MST spanning S
while |S| < |V| do
    find a minimum weight edge (u, v):  u ∈ S and v ∈ V − S
    S ← S + v
    T ← T + (u, v)
return T
```

Running time analysis:

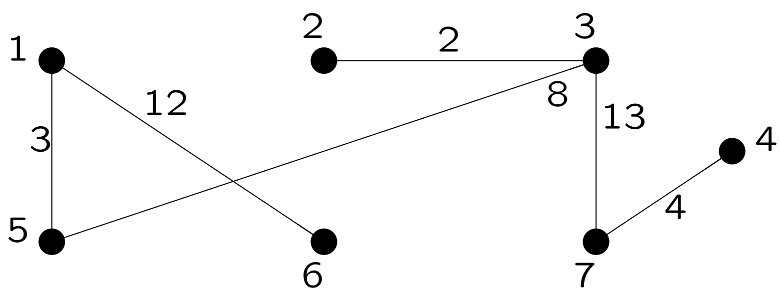
1. finding such an edge in $O(n^2)$ (or $O(m)$) time
2. there are $n - 1$ edges in the output MST
3. therefore, in total $O(n^3)$ (or $O(nm)$) time

Prim's algorithm for the MST problem — an example:

- Input graph G:



- $\texttt{primMST}(G, w, 1)$ returns:



- Correctness of Prim's algorithm (next)

- Improvement over the simplest implementation

  Observation: every iteration it looks for minimum weight edge

  — heap might help (next lecture)

Prim's algorithm for the MST problem — correctness:

- Input graph $G = (V, E)$: $E = \{e_1, e_2, \ldots, e_m\}$
- Suppose edges in the output tree T are $e_{i_1}, e_{i_2}, \ldots, e_{i_{n-1}}$ (in the order picked by Prim's algorithm)
- Want to prove: T is an MST
- Suppose T' is an MST and it contains edges $e_{j_1}, e_{j_2}, \ldots, e_{j_{n-1}}$ (sorted in the way that it maps the edge order in T as much as possible). If $T \neq T'$ (otherwise we are done), then
  - there is a minimum index k, such that $e_{j_k} \neq e_{i_k}$
  - let T_0 denote the tree formed by $\{e_{i_1}, e_{i_2}, \ldots, e_{i_{k-1}}\}$
  - let $V_0 = V(T_0)$ and $V_1 = V - V_0$
  - adding e_{i_k} to T' creates a cycle which contains some edge, say e_{j_p}, that has one ending vertex in V_0 and the other in V_1
  - $T'' = T' + e_{i_k} - e_{j_p}$ is another spanning tree
  - T'' is another MST (why ?) sharing one more edge with T
  - repeat this argument to claim that T is also an MST
- Note: this is a proof using 'contradiction' + 'graph theory'.
- Proof can also be done by

  (`while`) Loop Invariant: <u>T is an MST on S.</u>

  Exercise !

Have you understood the lecture contents?

well	ok	not-at-all	topic
☐	☐	☐	minimum spanning tree
☐	☐	☐	greedy algorithms in general
☐	☐	☐	Prim's algorithm: idea
☐	☐	☐	one simplest implementation
☐	☐	☐	execution, & analysis
☐	☐	☐	correctness