Lecture 28: Graph Algorithms

Agenda:

- Prim's MST algorithm
- Kruskal's MST algorithm

Reading:

- Textbook pages 561 – 579

Prim's algorithm for the MST problem (recall):

- Input: an edge-weighted (simple, undirected, connected) graph (positive weights)
- Output: an MST
- Idea:
  - suppose we have already an MST T' spanning subset V' of vertices (T' is initialized empty and V' is initialized to contain any one vertex)
  - grow T' to span one more vertex $v \in V - V'$
  - v is selected such that there is a vertex $u \in V'$, edge (u, v) is the minimum weighted over all edges of form (u', v') where $u' \in V'$ and $v' \in V - V'$
  - when V' becomes V, terminate
- One simplest implementation:

```
procedure primMST(G, w, r)          **G = (V, E)

S ← {r}                             **S: spanned vertex subset
T ← ∅                               **T: MST spanning S
while |S| < |V| do
    find a minimum weight edge (u, v):  u ∈ S and v ∈ V − S
    S ← S + v
    T ← T + (u, v)
return T
```

Running time analysis:

1. finding such an edge in $O(n^2)$ (or $O(m)$) time
2. there are $n - 1$ edges in the output MST
3. therefore, in total $O(n^3)$ (or $O(nm)$) time

Prim's algorithm for the MST problem — correctness:

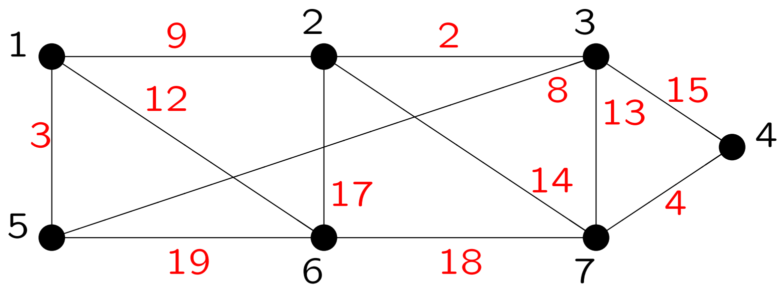
- Input graph $G = (V, E)$: $E = \{e_1, e_2, \ldots, e_m\}$
- Suppose edges in the output tree T are $e_{i_1}, e_{i_2}, \ldots, e_{i_{n-1}}$ (in the order picked by Prim's algorithm)
- Want to prove: T is an MST
- Suppose T' is an MST and it contains edges $e_{j_1}, e_{j_2}, \ldots, e_{j_{n-1}}$ (sorted in the way that it maps the edge order in T as much as possible). If $T \neq T'$ (otherwise we are done), then
  - there is a minimum index k, such that $e_{j_k} \neq e_{i_k}$
  - let T_0 denote the tree formed by $\{e_{i_1}, e_{i_2}, \ldots, e_{i_{k-1}}\}$
  - let $V_0 = V(T_0)$ and $V_1 = V - V_0$
  - adding e_{i_k} to T' creates a cycle which contains some edge, say e_{j_p}, that has one ending vertex in V_0 and the other in V_1
  - $T'' = T' + e_{i_k} - e_{j_p}$ is another spanning tree
  - T'' is another MST (why ?) sharing one more edge with T
  - repeat this argument to claim that T is also an MST
- Note: this is a proof using 'contradiction' + 'graph theory'.
- Proof can also be done by

  (`while`) Loop Invariant: <u>T is an MST on S.</u>

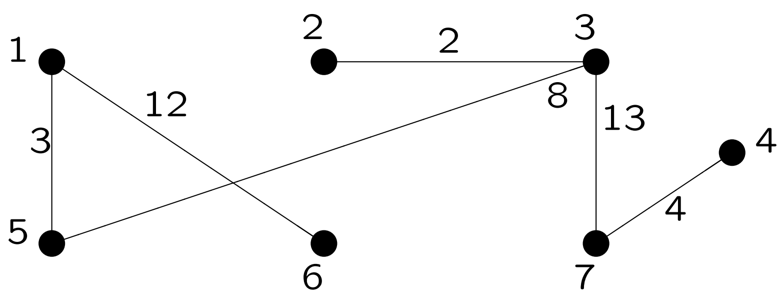
  Exercise !

Prim's algorithm for the MST problem — improvement:

- Where to improve: finding the minimum weight edge (u, v)
- Initially we need to scan all the edges, $\Theta(m)$ (worst case)
- Example: input graph G:

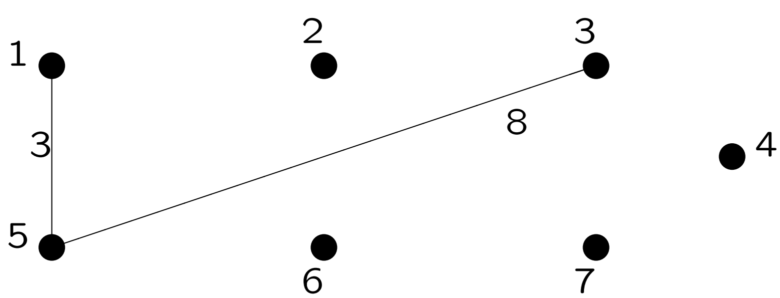


- `primMST`$(G, w, 1)$ returns:



- `primMST`$(G, w, 1)$: an intermediate tree

  What are the candidate edges ?



Prim's algorithm for the MST problem — improvement:

- Ideas:
  1. for each non-tree vertex v, store its minimum-weight tree neighbor $p[v]$
  2. store edges of type $(p[v], v])$ in a min-priority queue Q
  3. therefore, every time the target edge can be extracted $\texttt{ExtractMin}(Q)$

     note: need to update the neighbor information for non-tree vertices after the extraction

- Pseudocode:

```
procedure primMST(G, w, r)                    **G = (V, E)

for each v ∈ V(G) do
    key[v] ← ∞
    p[v] ← NIL
key[r] ← 0
Q ← V(G)
while Q ≠ ∅ do
    u ← ExtractMin(Q)                         **r dequeued first
    for each v ∈ Adj[u] do
        if v ∈ Q && w(u,v) < key[v] then      **update v
            p[v] ← u
            decrease-key(v, w(u,v))           **key[v] ← w(u,v)
```

- Analysis:
  - correctness (almost done — need to prove that ExtractMin(Q) does extract the minimum weight edge)
  - running time: $\Theta\Big(n \log n + \sum_{u \in V} \big(\texttt{degree}(u) \times \log n\big)\Big)$

    so: $\Theta(m \log n)$ — adjacency list graph representation

Kruskal's algorithm for the MST problem:

- Input: an edge-weighted (simple, undirected, connected) graph (positive weights)
- Output: an MST
- Idea:
  - suppose we have already an acyclic subgraph T
  - grow T to by including one more edge e
  - edge e is selected such that
    1. maintaining 'acyclic'
    2. of minimum weight
  - when T contains $n-1$ edges (and thus becomes a spanning tree), terminate
- One implementation using DSUF:

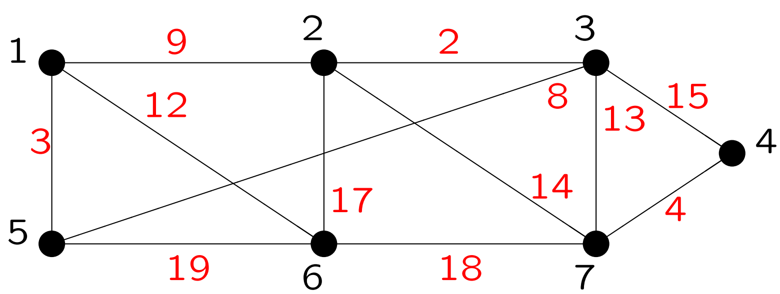
```
procedure kruskalMST(G, w)        **G = (V, E)

T ← ∅                             **T:  (V, T) acyclic
for each v ∈ V(G) do
    MakeSet(v)
sort edges in E(G) into non-decreasing weight order
for each (u, v) ∈ E(G) do         **taken in order
    if cFind(u) ≠ cFind(v) then   **(u, v) is 'safe'
    T ← T + (u, v)
    rUnion(u, v)
return (V, T)
```

Kruskal's algorithm for the MST problem — analysis:

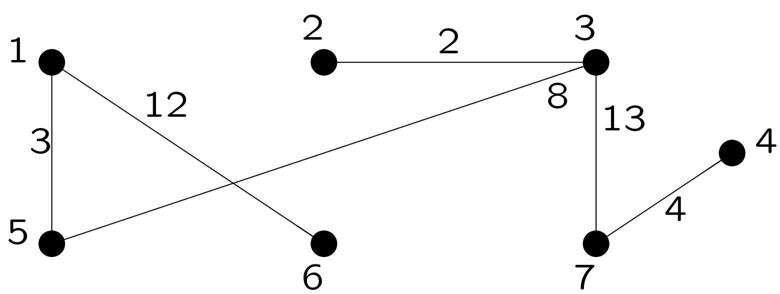
- Correctness:
  - by Loop Invariant: what is the loop invariant ???
  - by 'contradiction' + 'graph theory'

- Running time analysis (adjacency list graph representation):
  1. sorting edges $\Theta(m \log n)$
  2. $\Theta(m)$ **rUnion/cFind**: $O(m\alpha(n))$
  3. therefore, in total $\Theta(m \log n)$ time

- An example:



- **kruskalMST**(G, w) returns:



Have you understood the lecture contents?

well	ok	not-at-all	topic
☐	☐	☐	Prim's algorithm: correctness
☐	☐	☐	improved implementation and analysis
☐	☐	☐	Kruskal's algorithm: idea
☐	☐	☐	correctness
☐	☐	☐	DSUF implementation, execution, & analysis