Lecture 29: Graph Algorithms

Agenda:

- Single-source shortest paths
- Dijkstra's algorithm for non-negatively weighted case

Reading:

- Textbook pages 580 – 587, 595 – 601

Shortest path problems:

- BFS recall: outputs every s-to-v shortest path
  - s — start vertex
  - v — reachable vertex from s (residing in a same connected component)
  - shortest — # edges
  - running time $\Theta(n + m)$
- BFS solves *the single-source-shortest-path problem* on undirected unweighted graphs

  Single-Source-Shortest-Path (SSSP) problem: given a source s, find out for all vertices their shortest paths from s
- Variants:
  - single source *vs.* all pairs
  - graphs: undirected *vs.* directed
  - edges: unweighted *vs.* weighted
  - edge weights: non-negative *vs.* may have negative weights
  - digraphs: acyclic *vs.* may have di-cycles

  Note: if there is no path, the distance is set to ∞ ...
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  1. SSSP problem on non-negatively weighted digraphs
     Dijkstra's algorithm (today)
  2. SSSP problem on weighted digraphs
     Bellman-Ford's algorithm (next lecture)

Dijkstra's SSSP algorithm:

- $d[v]$ — weight of the shortest path from source s to v
  if no such path, set to ∞
- Idea in Dijkstra's algorithm:
  - greedily grows an SSSP tree
  - ensures that when adding a vertex, its shortest path in the current (induced) subgraph is determined
  - records for every non-tree vertex v its best parent tree vertex $p[v]$

  Note: very similar to Prim's MST algorithm (the min-priority queue implementation)
- Pseudocode (use $d[v]$ as the key):

```
procedure dijkstra(G, w, s)        **G = (V, E)

for each v ∈ V(G) do               **initialization
    d[v] ← ∞
    p[v] ← NIL
d[s] ← 0
Q ← V(G)
while Q ≠ ∅ do
    u ← ExtractMin(Q)              **s dequeued first
    for each v ∈ Adj[u] do
        if d[u] + w(u,v) < d[v] then
                                   **update v, no matter if v ∈ Q
            p[v] ← u
            decrease-key(v, d[u] + w(u,v))
                                   **d[v] ← d[u] + w(u,v)
```

Dijkstra's SSSP algorithm *vs.* Prim's MST algorithm:

- procedure primMST(G, w, r) **$G = (V, E)$

```
for each v ∈ V(G) do                 **initialization
    key[v] ← ∞
    p[v] ← NIL
key[r] ← 0
Q ← V(G)
while Q ≠ ∅ do
    u ← ExtractMin(Q)                **r dequeued first
    for each v ∈ Adj[u] do
        if v ∈ Q && w(u,v) < key[v] then
                                     **update v
            p[v] ← u
            decrease-key(v, w(u,v))
                                     **key[v] ← w(u,v)
```

- procedure dijkstra(G, w, s) **$G = (V, E)$

```
for each v ∈ V(G) do                 **initialization
    d[v] ← ∞
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            p[v] ← u
            decrease-key(v, d[u] + w(u,v))
                                     **d[v] ← d[u] + w(u,v)
```
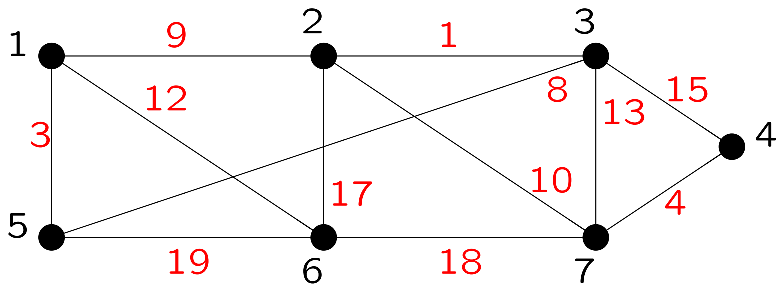
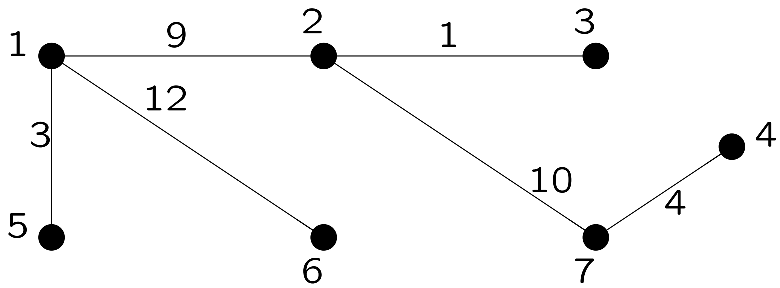

Dijkstra's SSSP algorithm — an example:

- Input graph G:



- `dijkstra`$(G, 1)$:



- `dijkstra`$(G, 1)$ trace:

v $d[v]/p[v]$	1 0/NIL	2 ∞/NIL	3 ∞/NIL	4 ∞/NIL	5 ∞/NIL	6 ∞/NIL	7 ∞/NIL
1 dequeued	0/NIL	9/1	∞/NIL	∞/NIL	3/1	12/1	∞/NIL
5 dequeued	0/NIL	9/1	11/5	∞/NIL	3/1	12/1	∞/NIL
2 dequeued	0/NIL	9/1	10/2	∞/NIL	3/1	12/1	19/2
3 dequeued	0/NIL	9/1	10/2	25/3	3/1	12/1	19/2
6 dequeued	0/NIL	9/1	10/2	25/3	3/1	12/1	19/2
7 dequeued	0/NIL	9/1	10/2	23/7	3/1	12/1	19/2

Dijkstra's SSSP algorithm — analysis:

- Applies to undirected graphs too

  See the last example :-)

- Running time:

  Same as the running time for Prim's MST algorithm

  — $\Theta(m \log n)$, assuming adjacency list graph representation and min-priority queue implemented by a heap

- Correctness:

  Let $S = V - Q$

  (`while`) Loop Invariant: for every $v \in S$, $d[v]$ records the weight of the shortest path from s to v in graph G

  Proof:

  - initialization (S is empty):
  - maintenance:
    Exercise: fill in the detail
  - termination: S becomes V, so LI implies that for every v, $d[v]$ records the weight of the shortest path from s to v in graph G

Dijkstra's SSSP algorithm — proof of maintenance:

- Maintenance (vertex u dequeued)

  $dist[u]$ — weight of a shortest path from s to u in G:

  — must show that at the end of loop body, $d[u] = dist[u]$

  Let $P = (s, v_1, v_2, \ldots, v_{k-1}, u)$ be any shortest path from s to u in graph G:

  - y — first vertex in P but not in S
  - x — the vertex before y in P
  - $dist[y] \le dist[u]$ — y on the path
  - $d[y] \ge d[u]$ — min-priority queue
  - $d[y] = dist[y]$ — since $x \in S$
  - conclusion: $d[u] \le dist[u]$

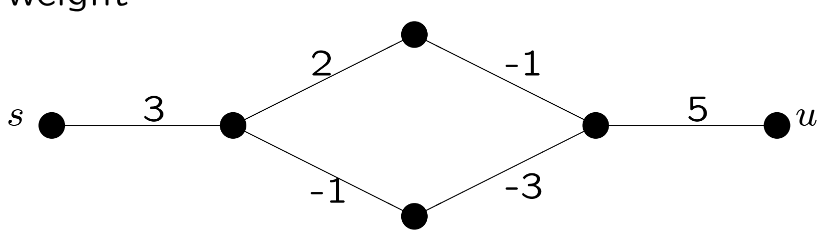
- Question: why Dijkstra's algorithm does NOT apply to negative weights?

  We fail to claim:

  — $dist[y] \le dist[u]$ — y on the path

- Another problem with negative weights:

  Suppose there is a (direct/undirected) cycle with a negative weight



  What is the weight of a shortest path from s to u ???

Have you understood the lecture contents?

well	ok	not-at-all	topic
☐	☐	☐	what is SSSP ???
☐	☐	☐	shortest path problem variants
☐	☐	☐	Dijkstra's algorithm: idea
☐	☐	☐	execution, correctness, & analysis