A Brief History and Recent Achievements in Bidirectional Search

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Key Related Work

- 1959 - Dijkstra’s Algorithm

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- 1966 - Bidirectional Search (Nicholson & Doran)

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- **1985 - A* Theory (Dechter & Pearl)**

Overview

- Bidirectional Theory
  - Eckerle et al, ICAPS 2017
- Optimal algorithm (offline)
  - Shaham et al, SoCS 2017
- Near-optimal algorithm (online)
  - Chen et al, IJCAI 2017

Assumptions

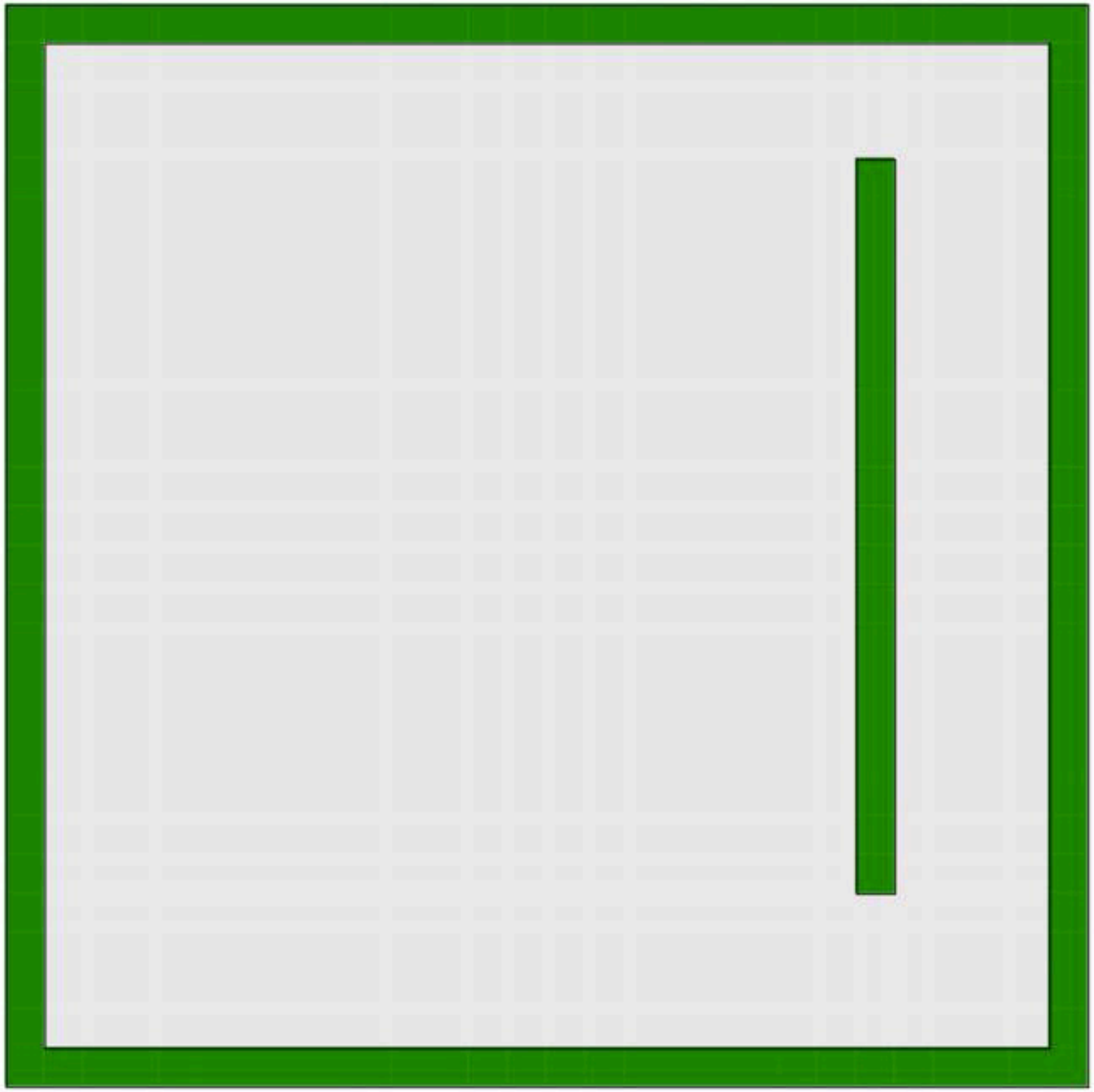
- Front-to-end bidirectional search
- Admissible algorithms
    - Performance with consistent heuristics
- Deterministic, black box algorithm

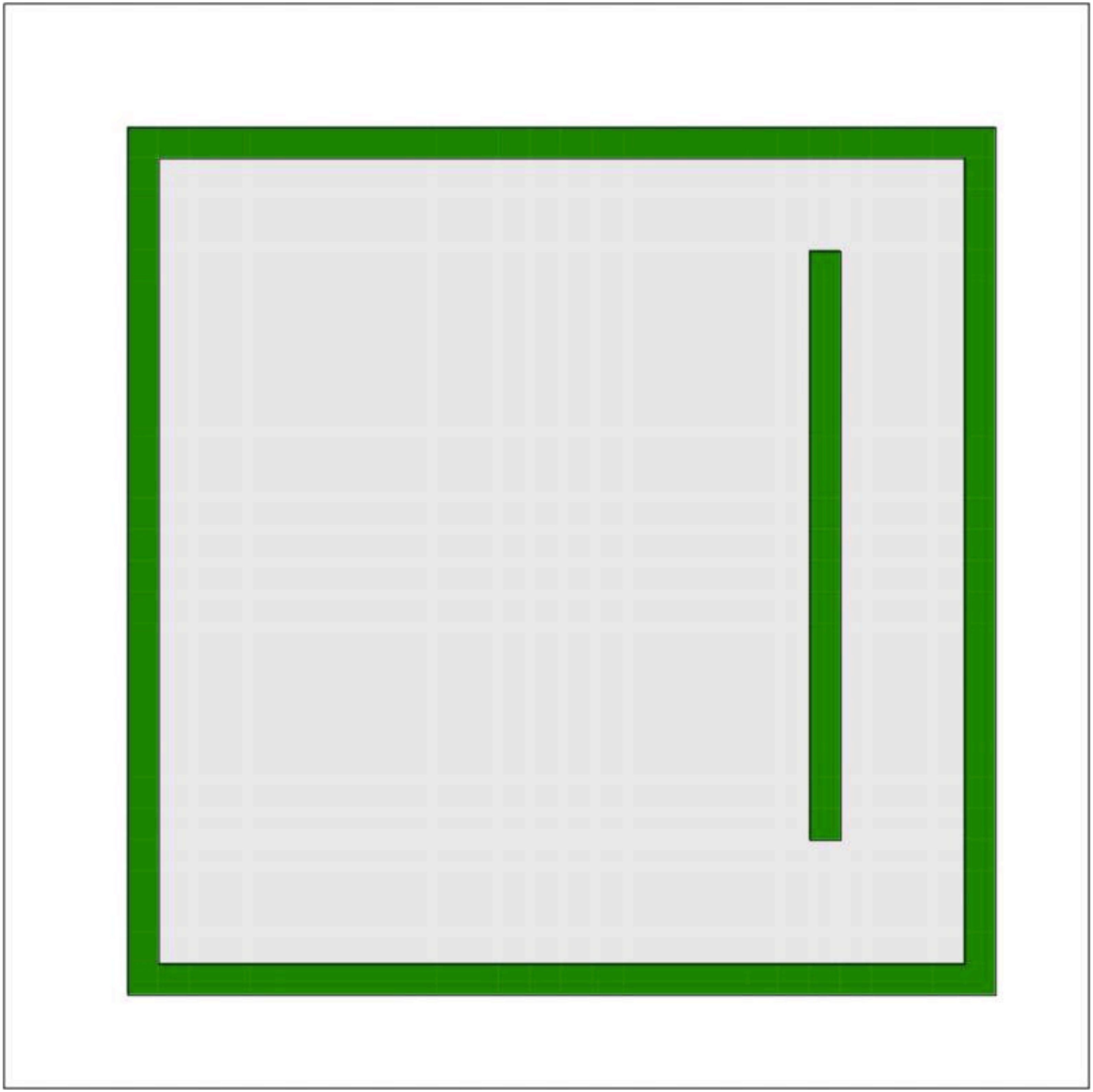
Unidirectional Theory

- ANY admissible unidirectional search algorithm:
  - Must expand ALL states with:
    - $f(s) = g(s) + h(s) < C^*$

- Otherwise we can construct instances on which it won't find the optimal solution

What states must be expanded by *all* bidirectional algorithms?





Conclusion

- Given a single state *s*
  - There exists a bidirectional algorithm that does not expand *s*

Conclusion

- Given a single state *s*
  - There exists a bidirectional algorithm that does not expand *s*
- Given some pairs of states (*u*, *v*)
  - We can avoid expanding *u*
  - We can avoid expanding *v*
  - We can't avoid expanding BOTH *u* and *v*

High-Level Picture

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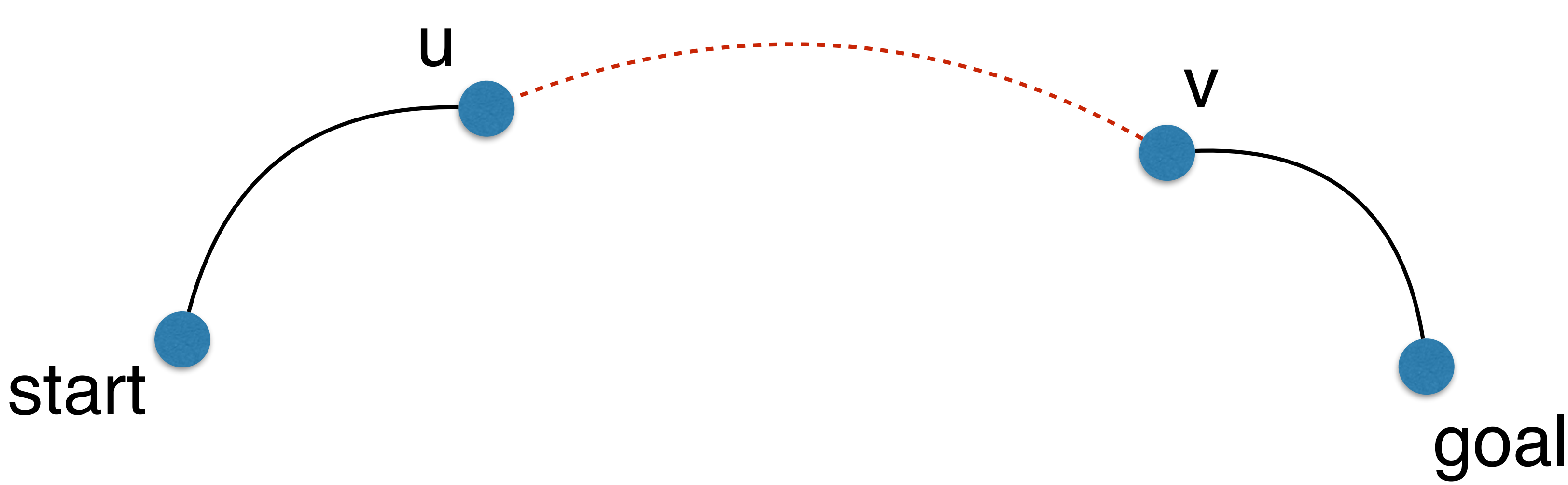
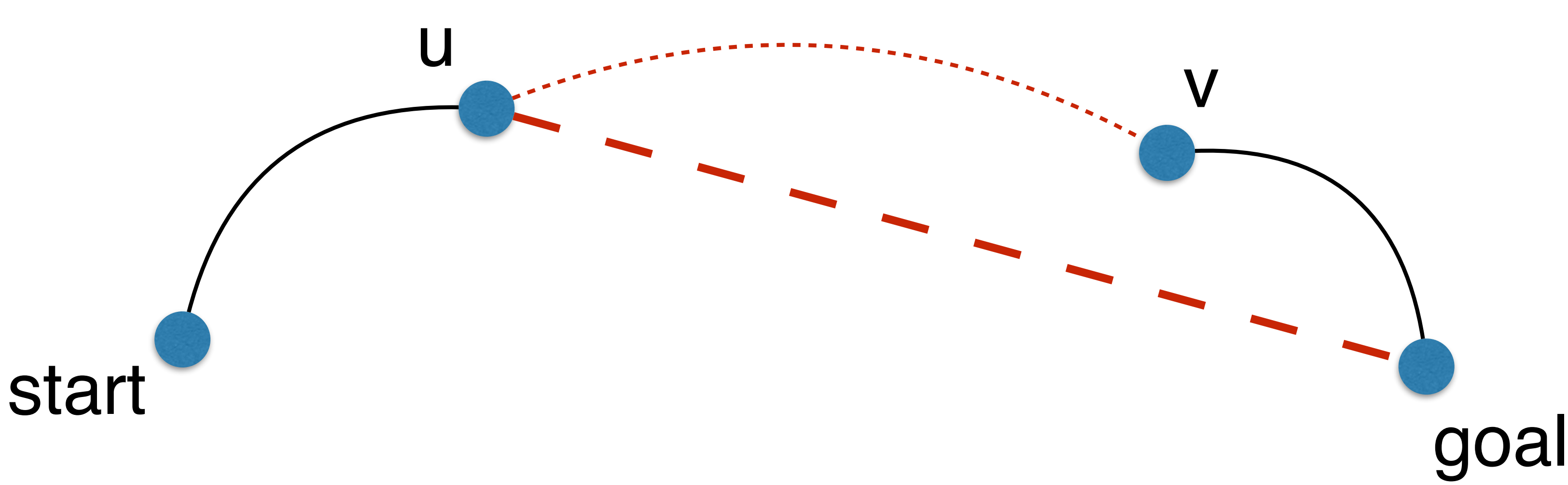


High-Level Picture



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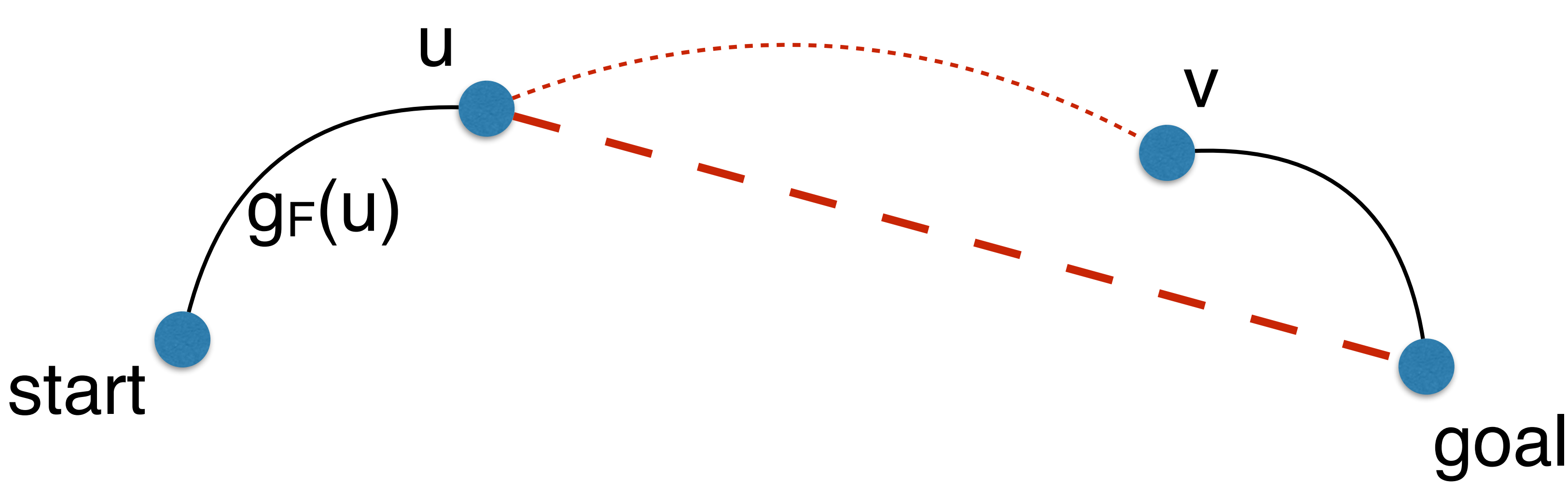
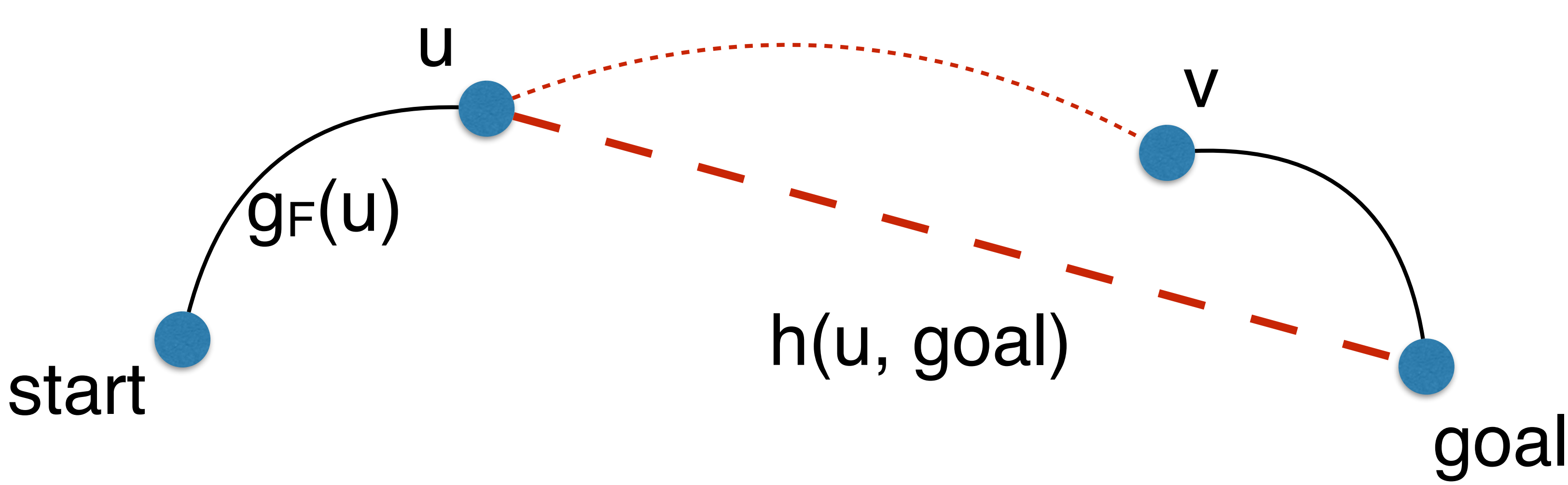
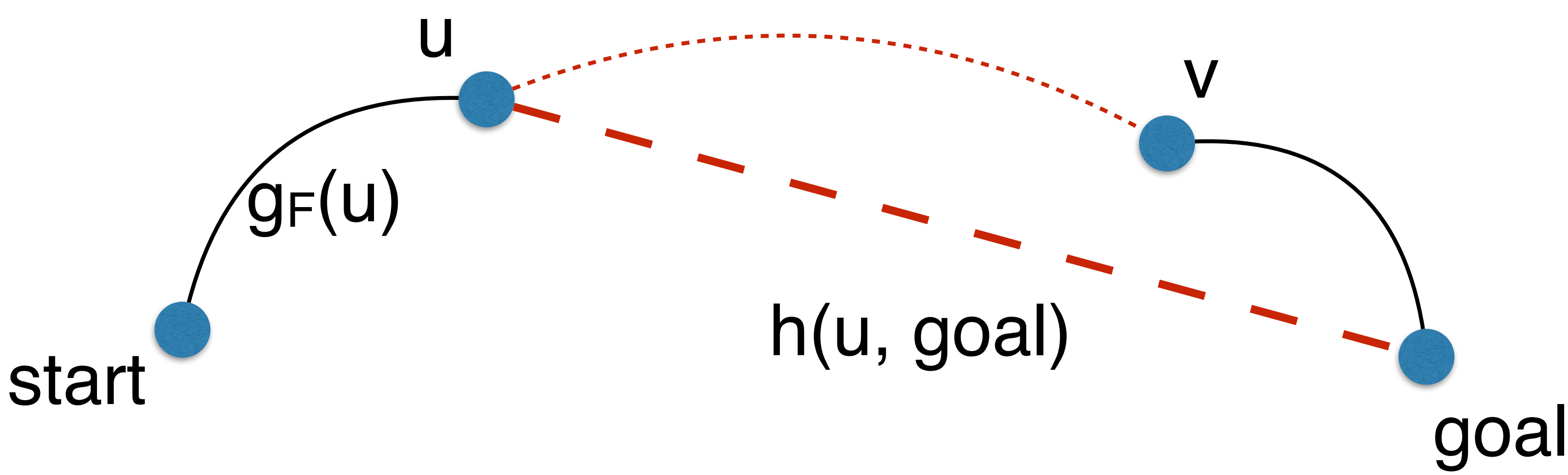


High-Level Picture

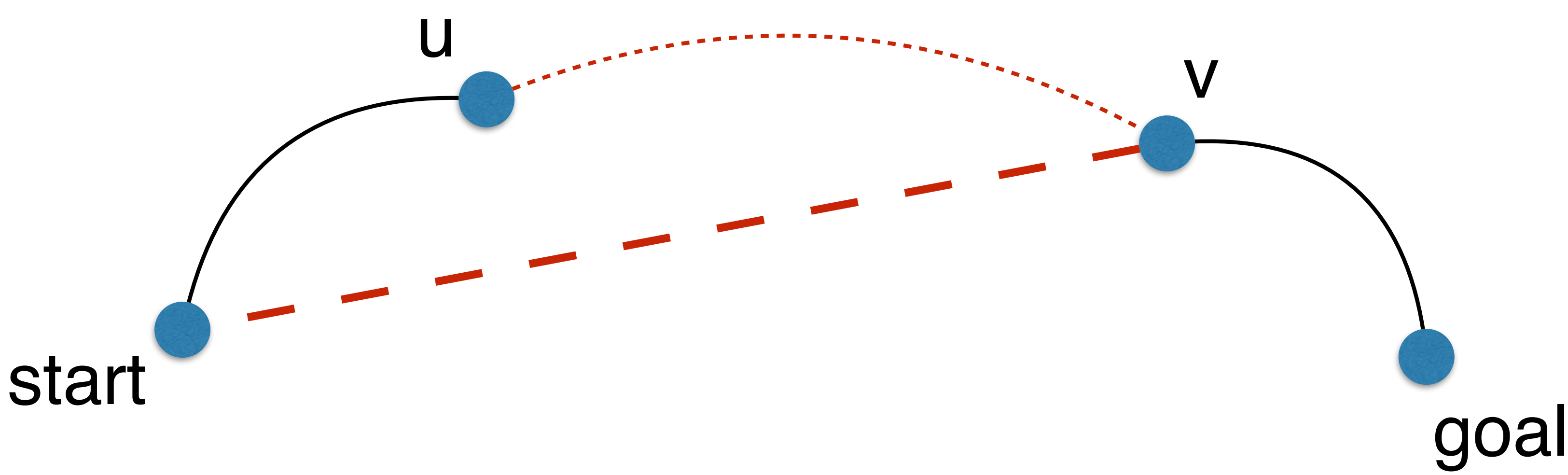


High-Level Picture



$$f_F(u) = g_F(u) + h(u, \text{goal})$$

High-Level Picture



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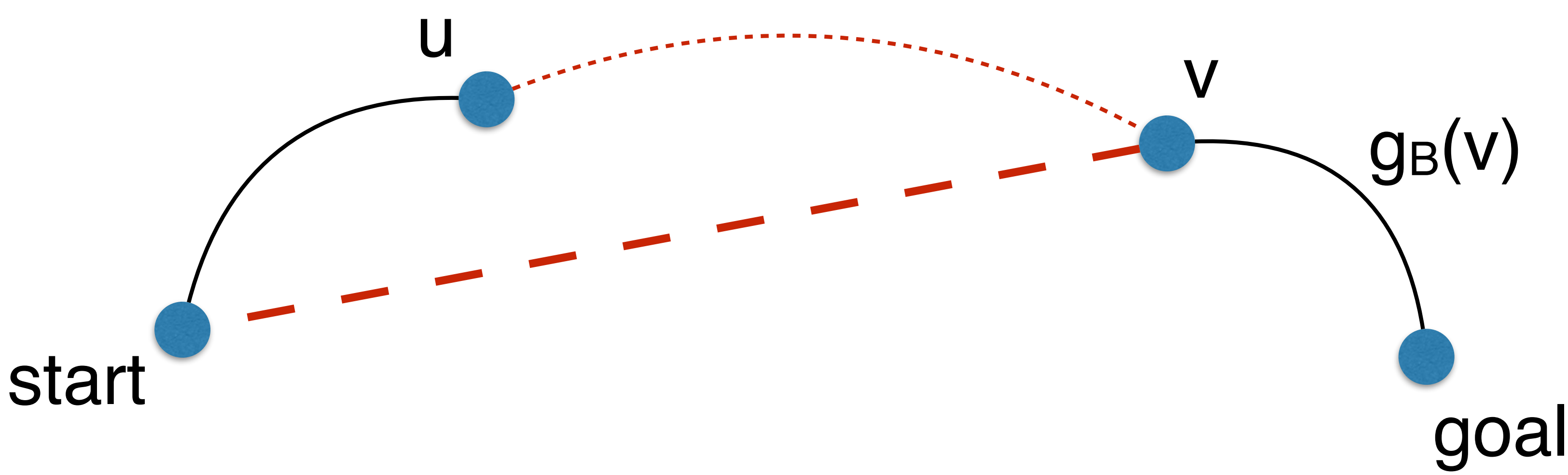

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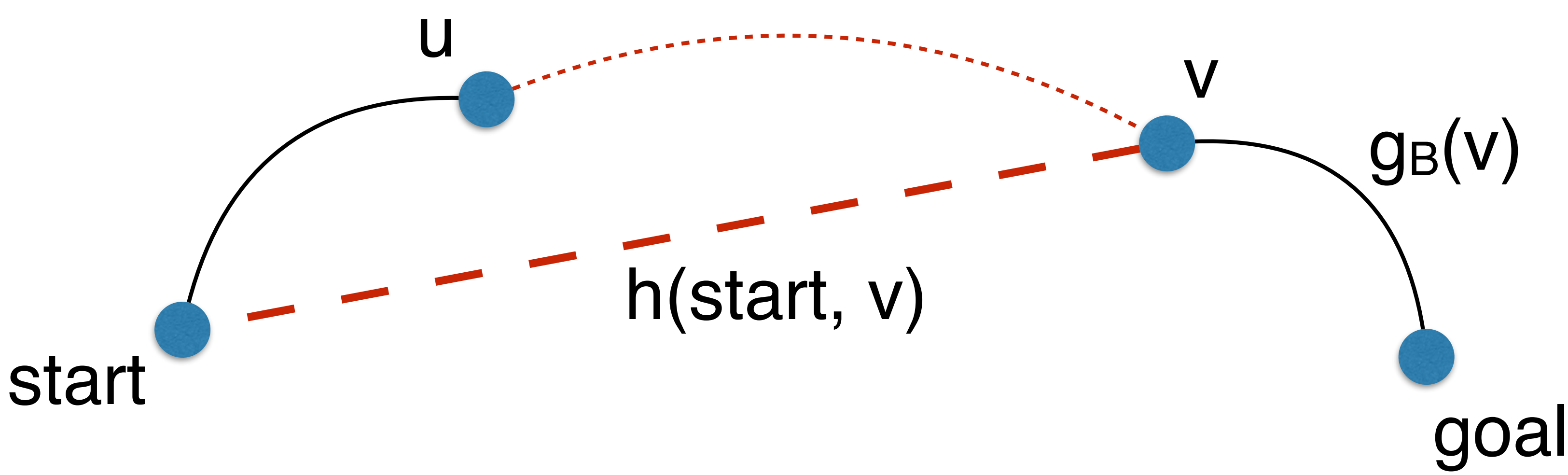

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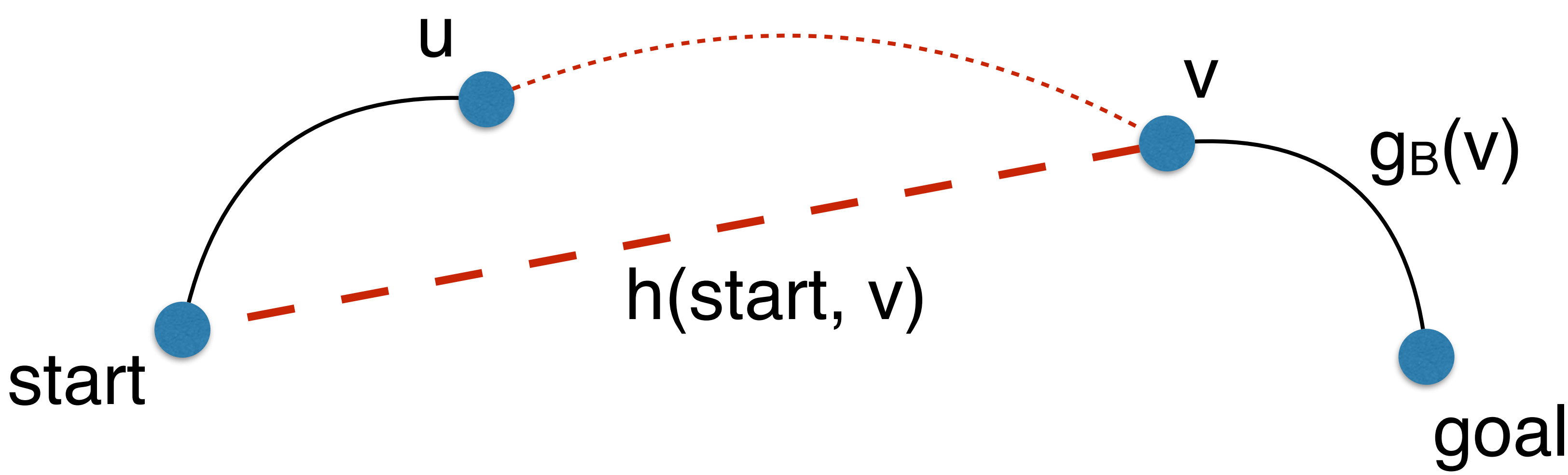
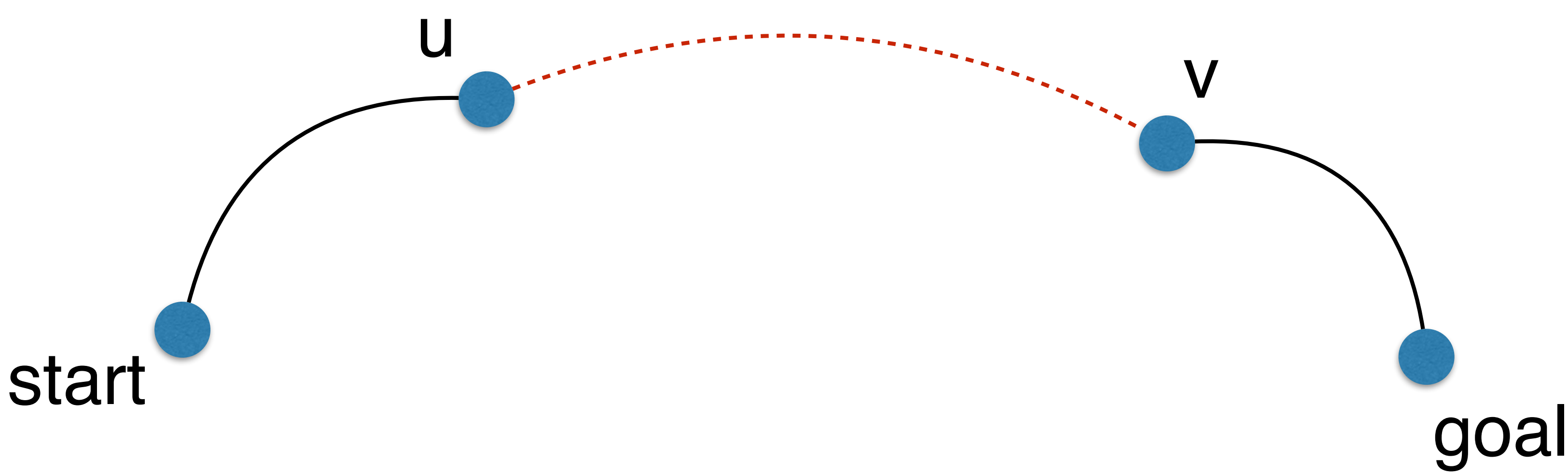


$$f_B(v) = g_B(v) + h(\text{start}, v)$$

High-Level Picture



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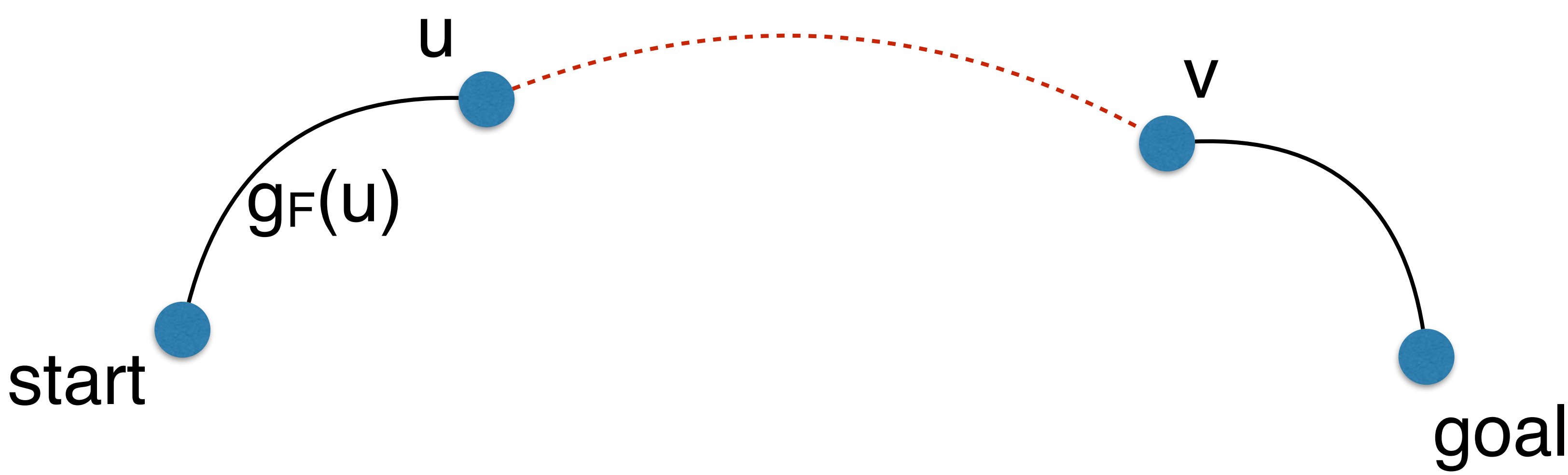

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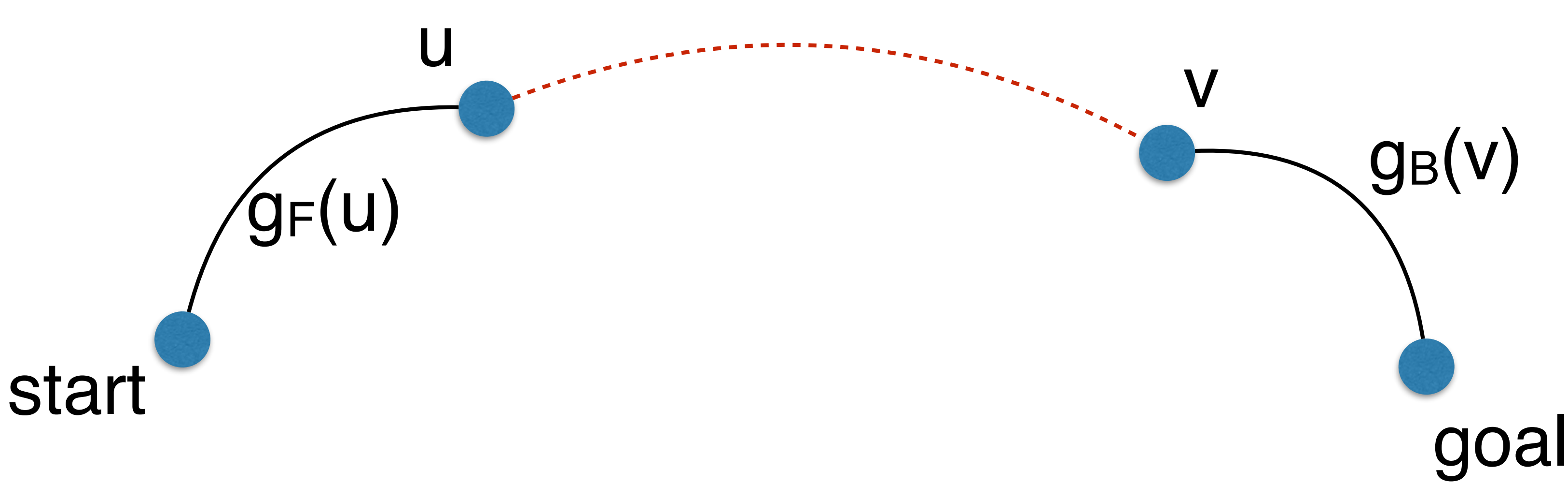

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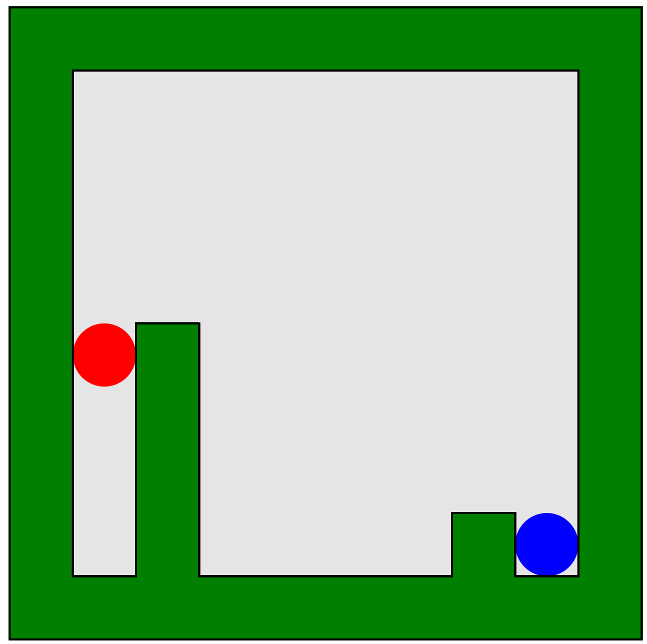
Theorem

f-cost:
estimate of total path length

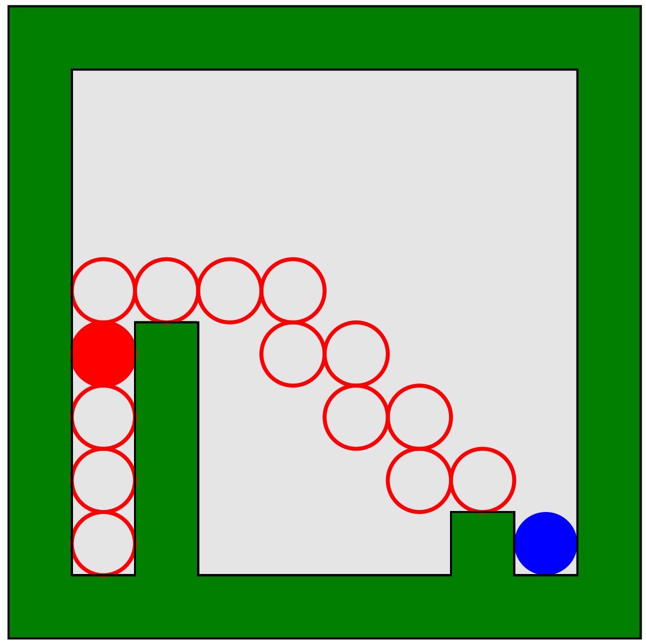
$$\begin{aligned}\mathrm{lb}(u,v) = max(&f_F(u),\\ &f_B(v),\\ &g_F(u) + g_B(v))\end{aligned}$$

- *If lb(u, v) < C* then we must expand either u or v*
- *Leads implicitly to termination conditions*

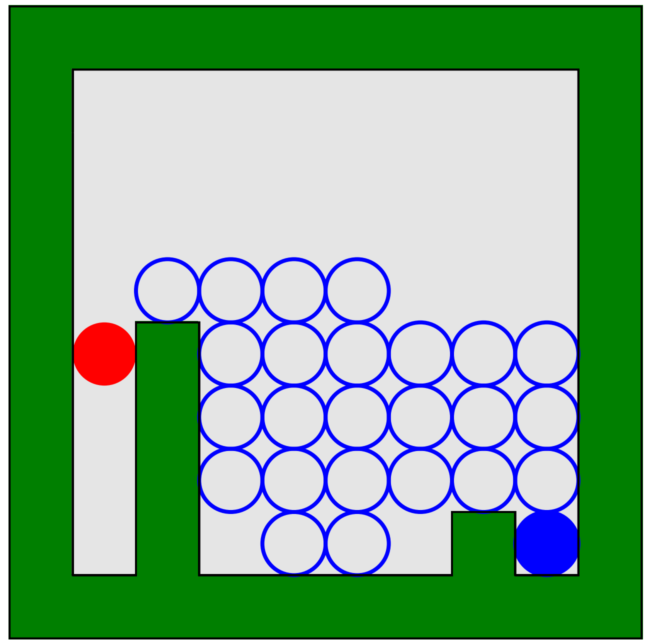
Sufficient Conditions for Node Expansion in Bidirectional Heuristic Search, Jurgen Eckerle, Jingwei Chen, Nathan Sturtevant, Sandra Zilles and Robert Holte, International Conference on Automated Planning and Scheduling (ICAPS), **2017**

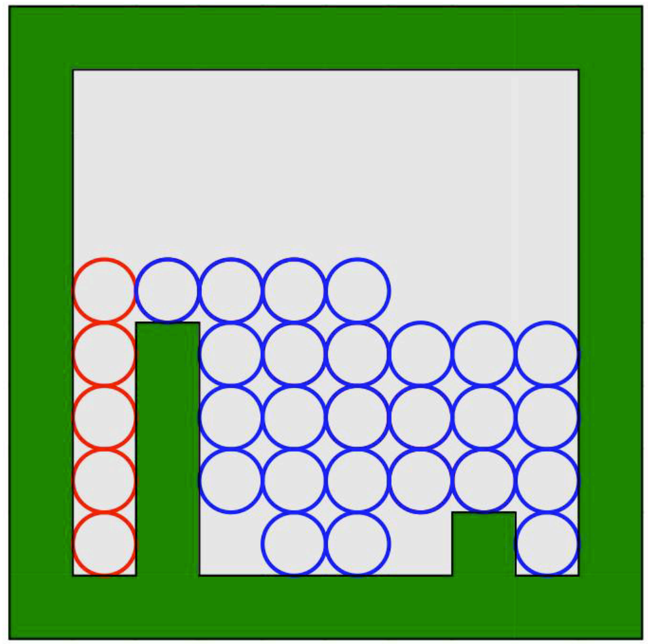


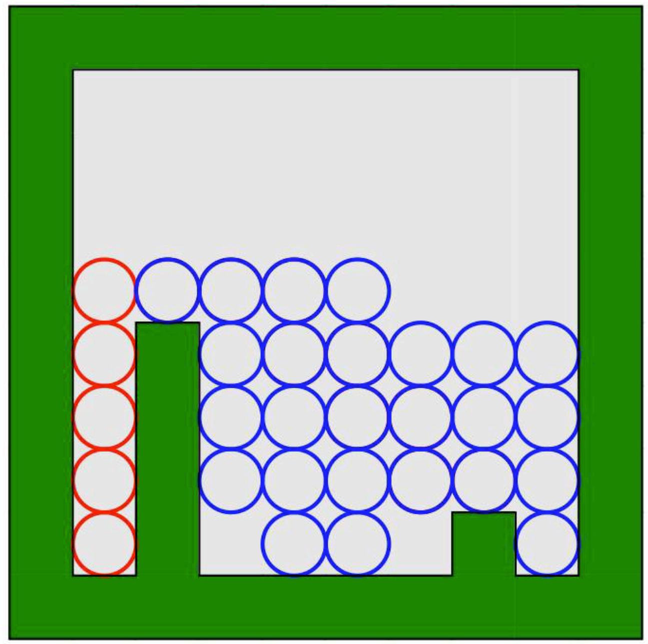
$$f_F(u) < C^*$$



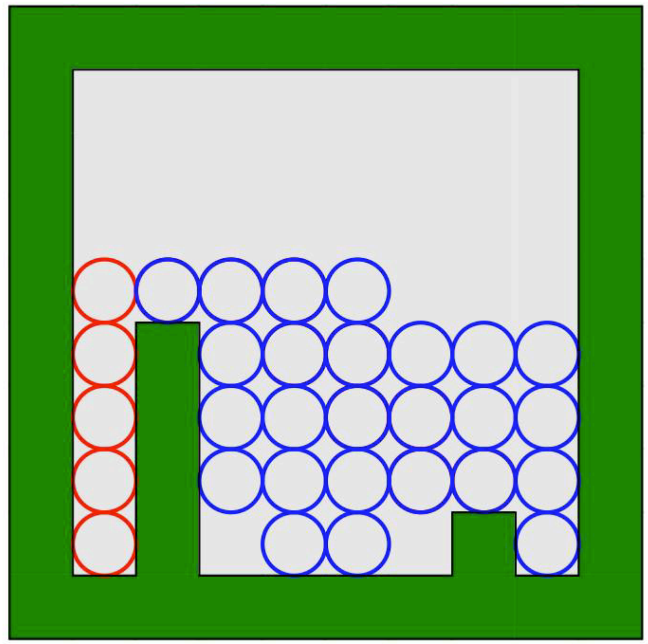
$$f_B(v) < C^*$$







$$g_F(u) + g_B(v) < C^*$$



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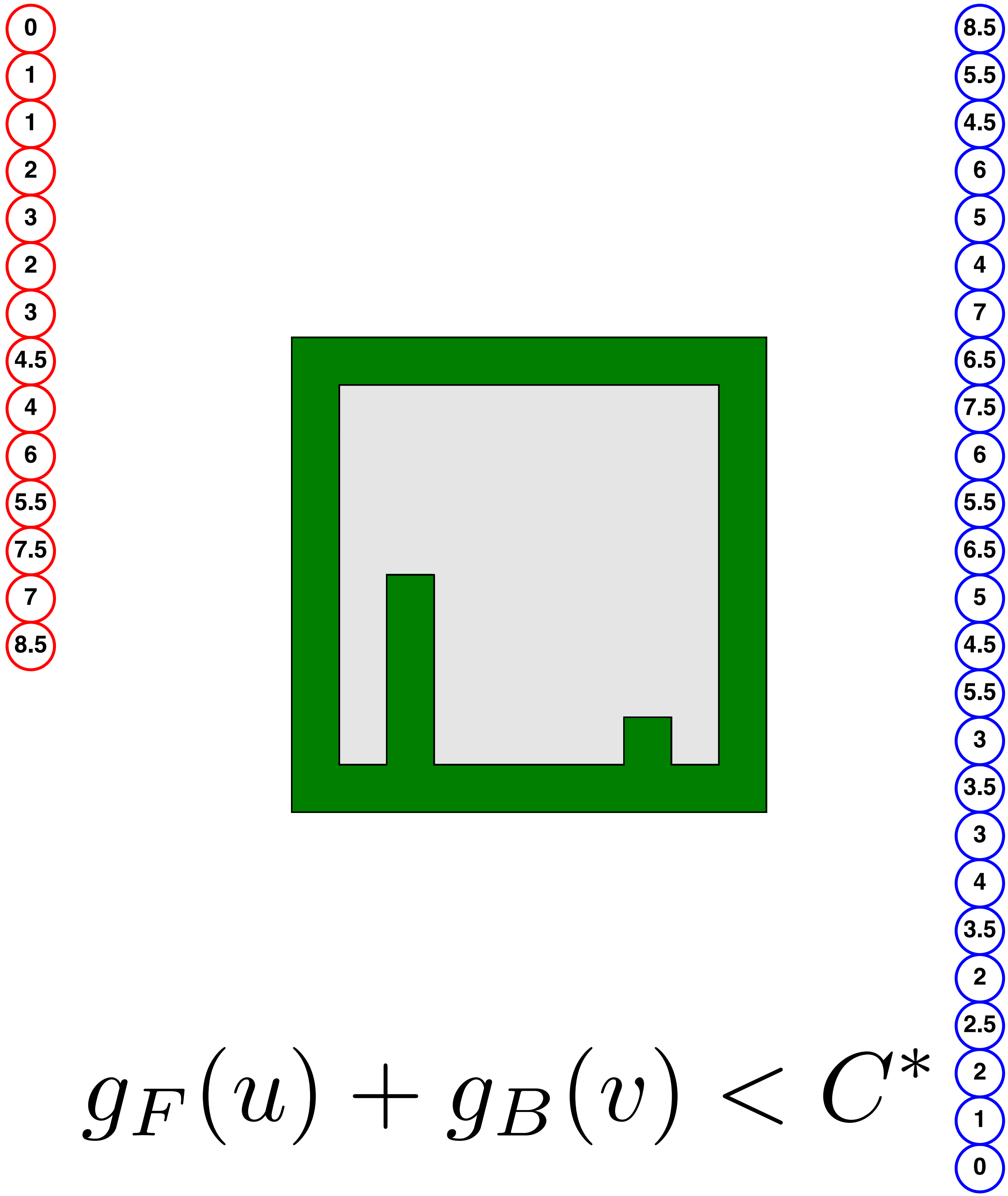
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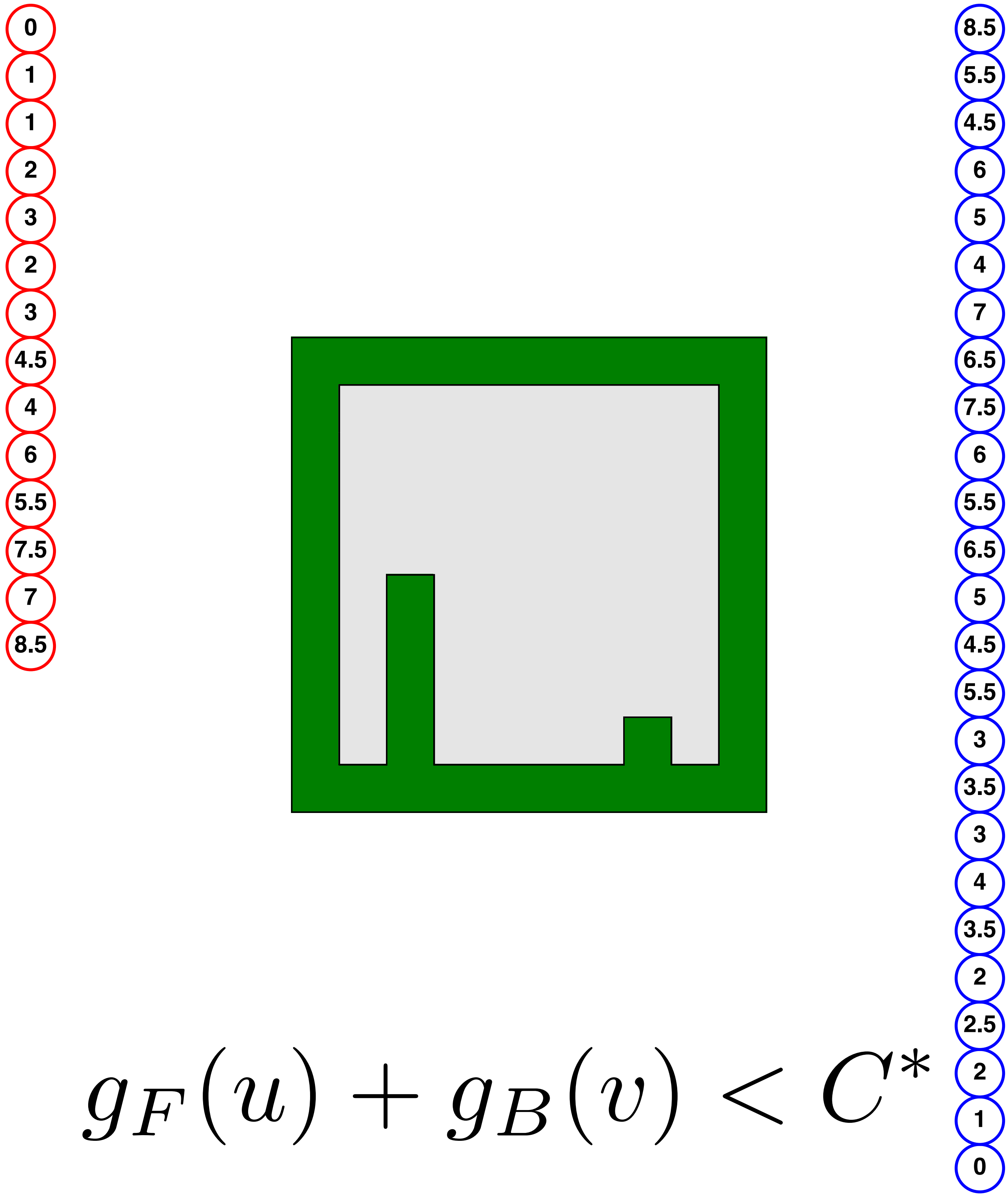
$$g_F(u) + g_B(v) < C^*$$

C* = 10.5



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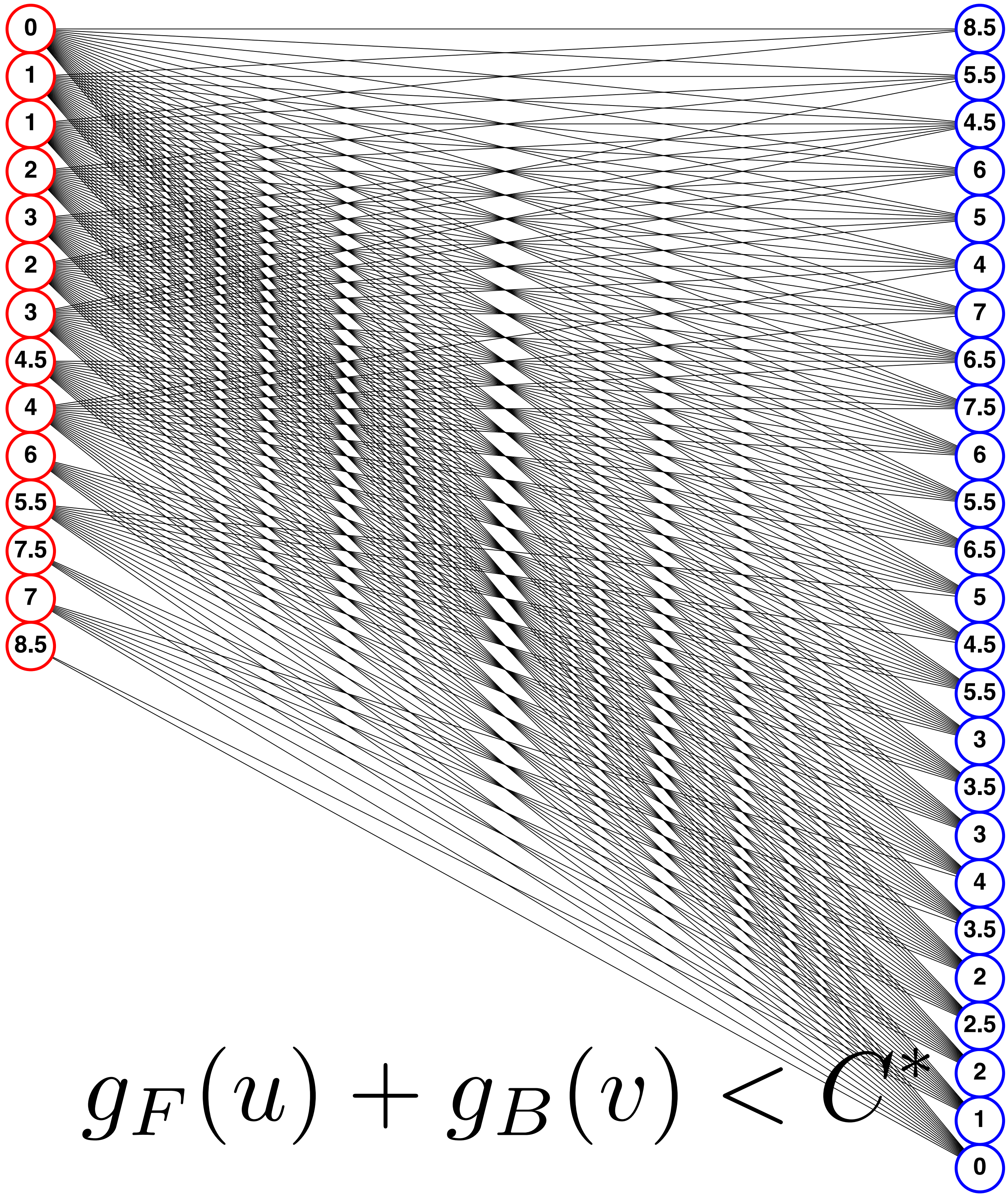

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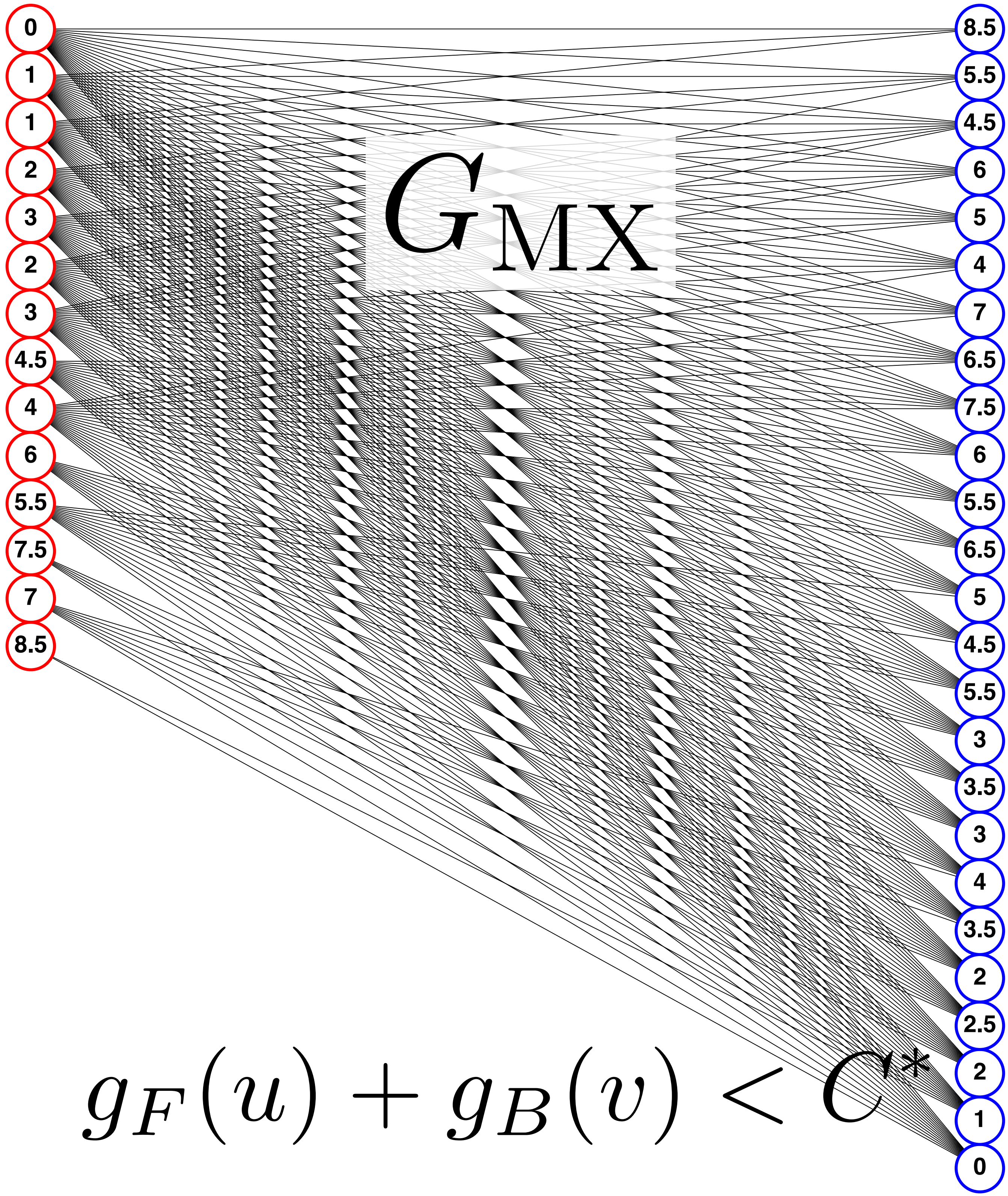

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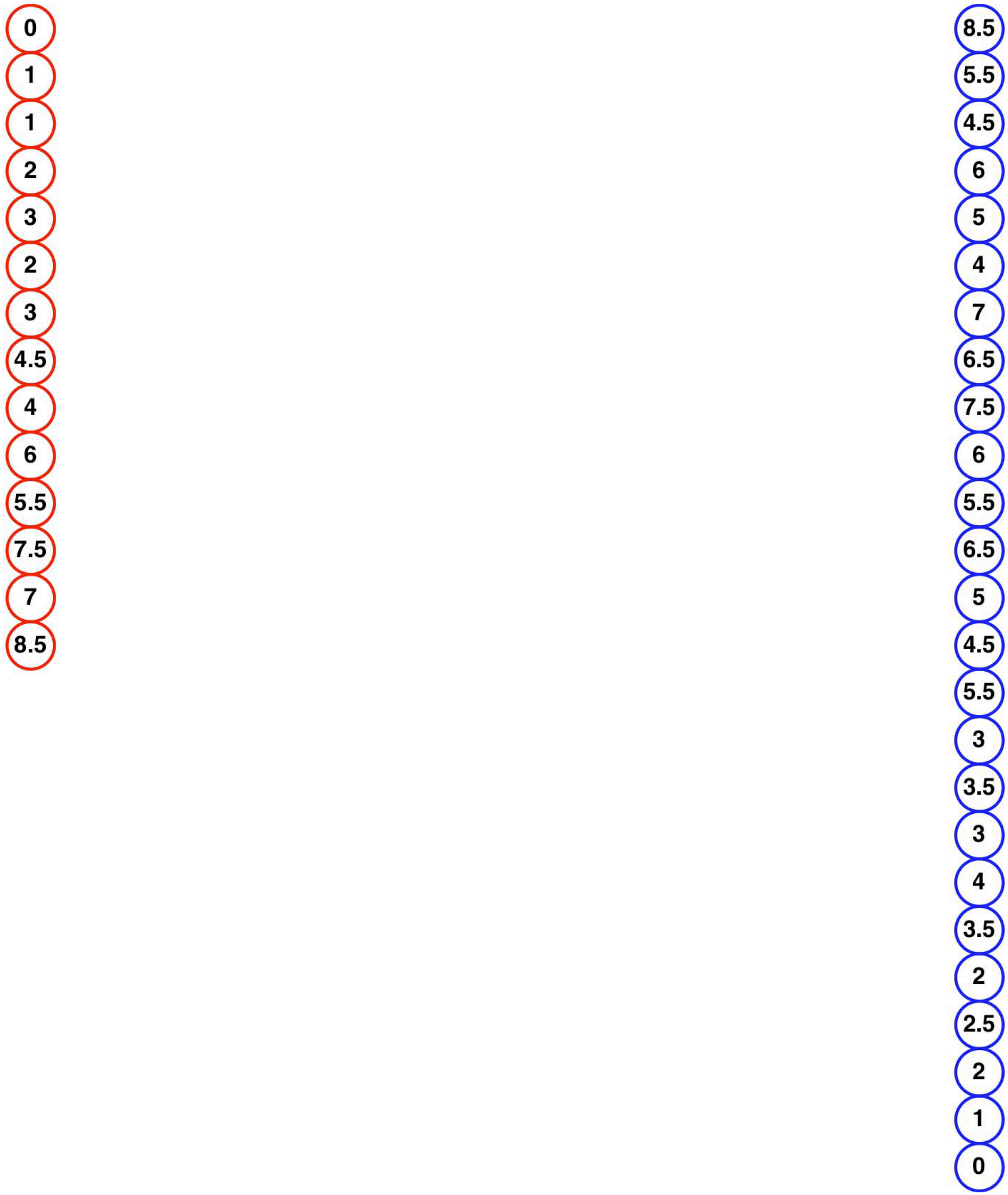
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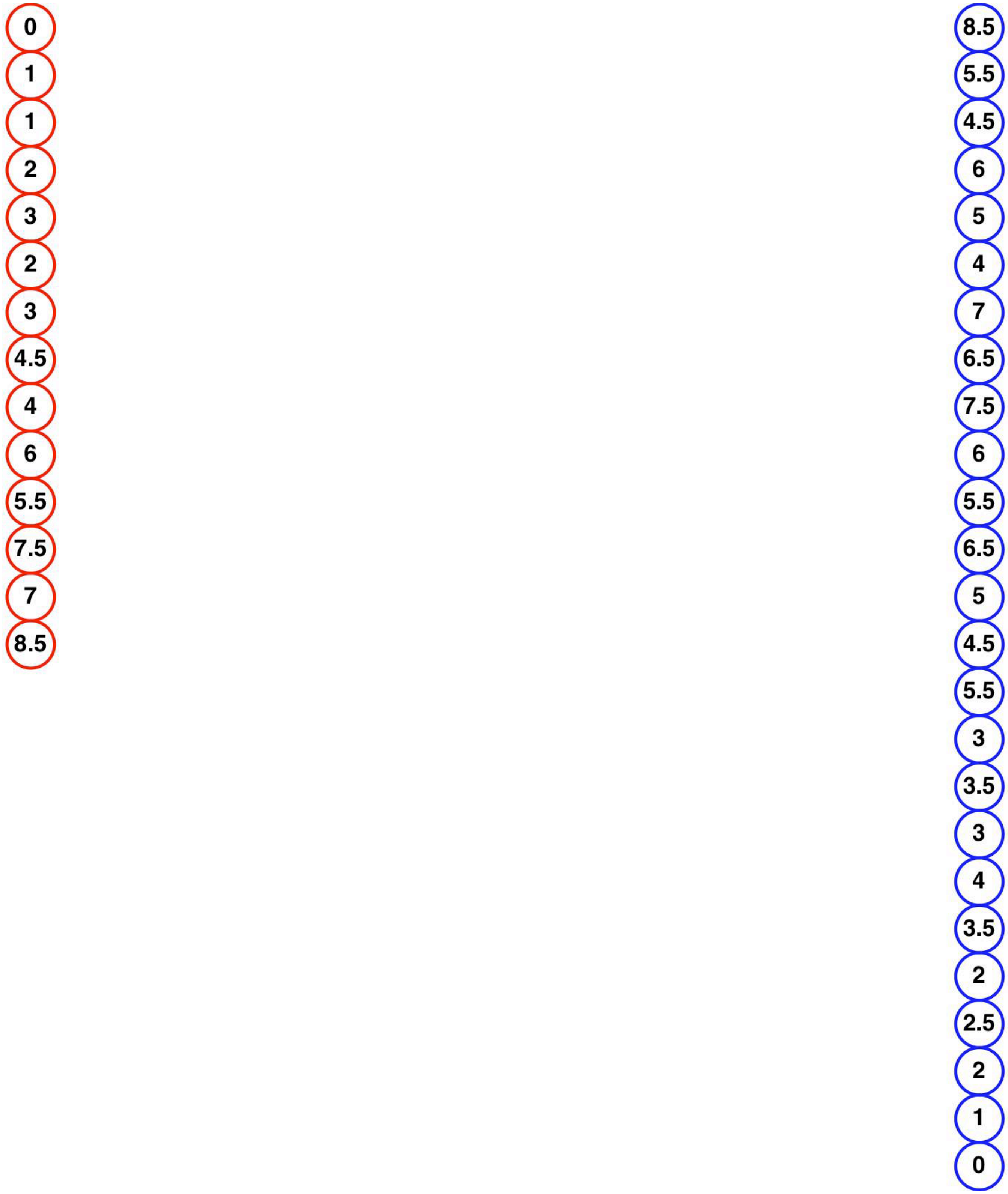
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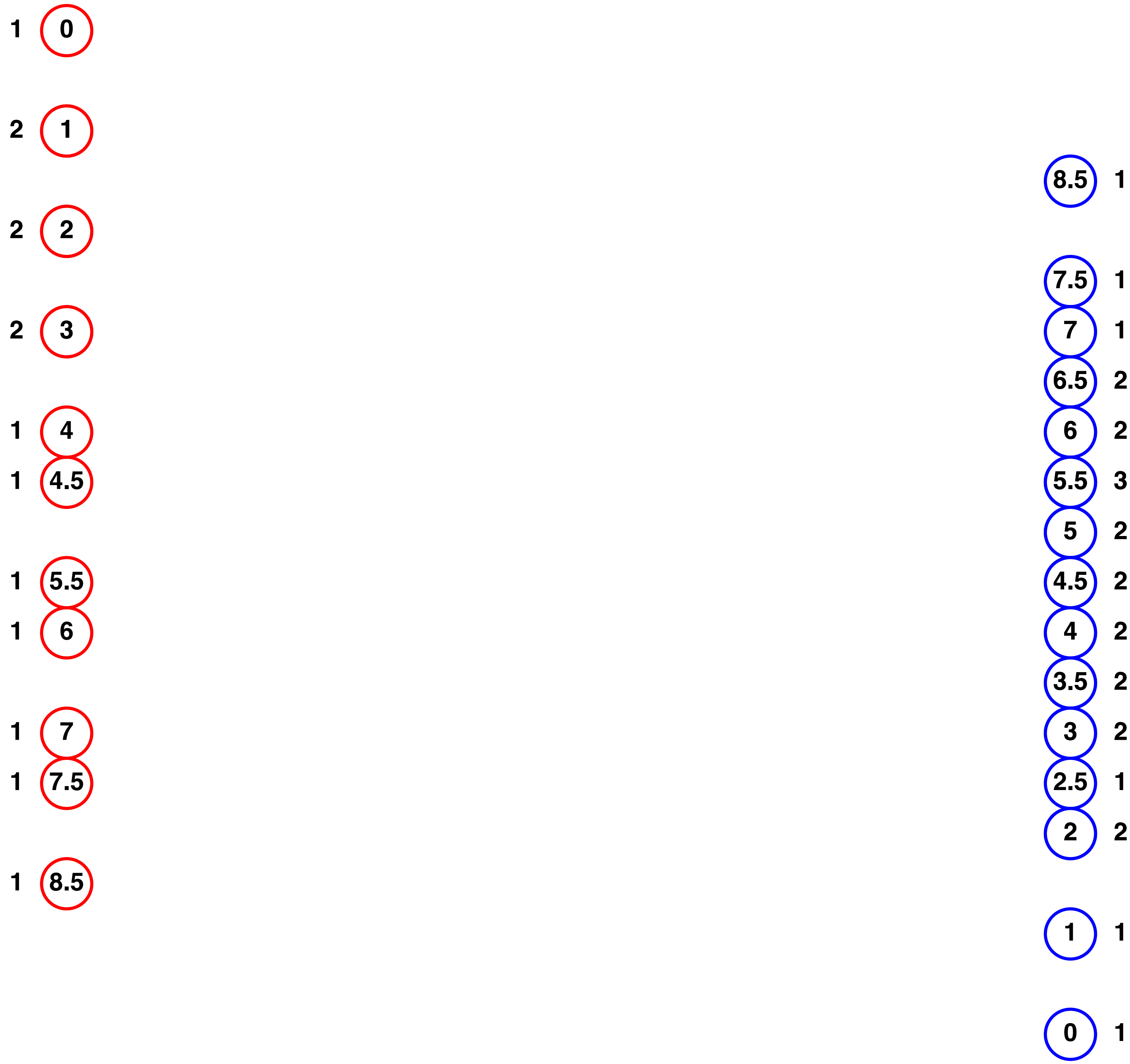

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1 7.5
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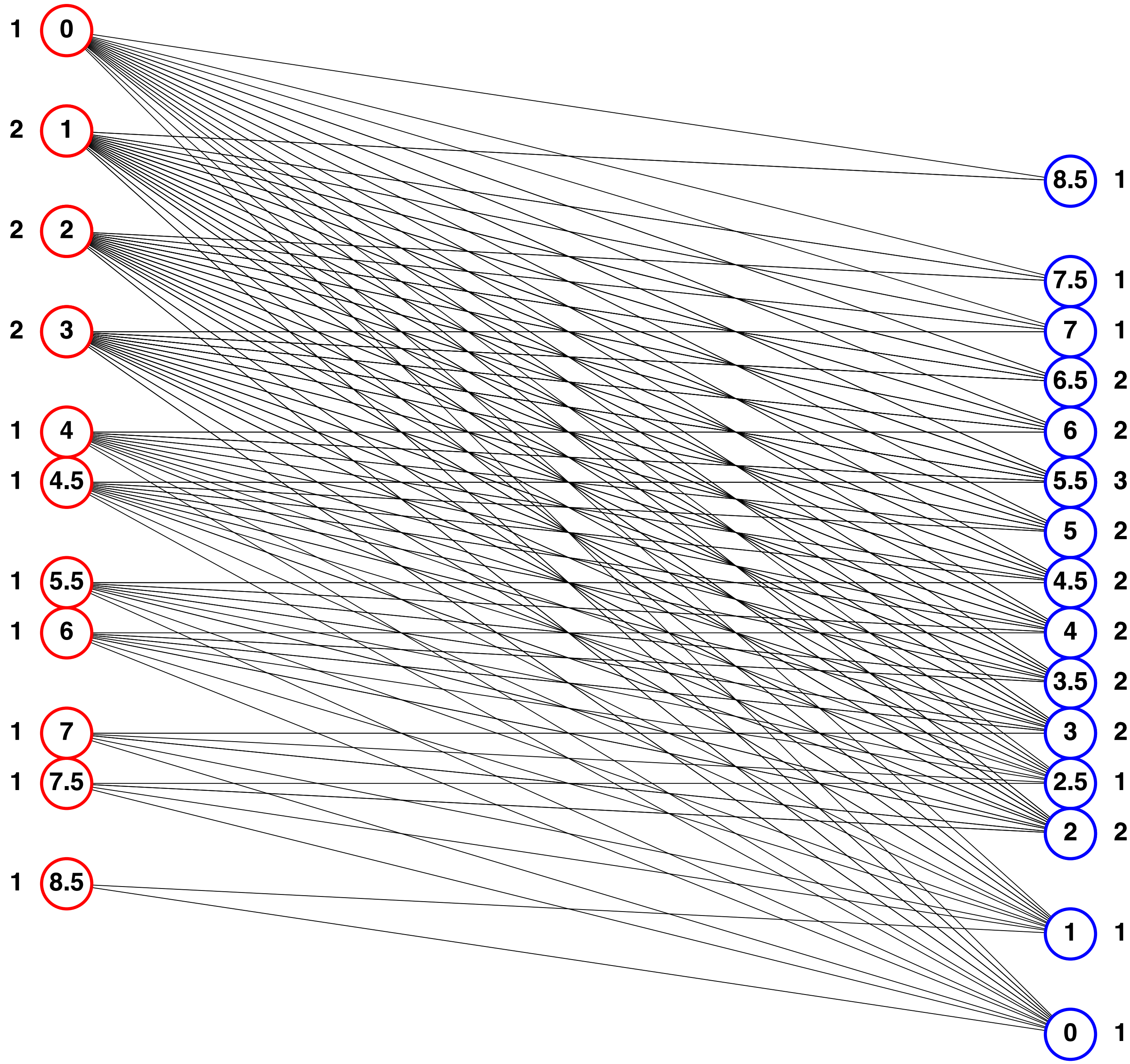

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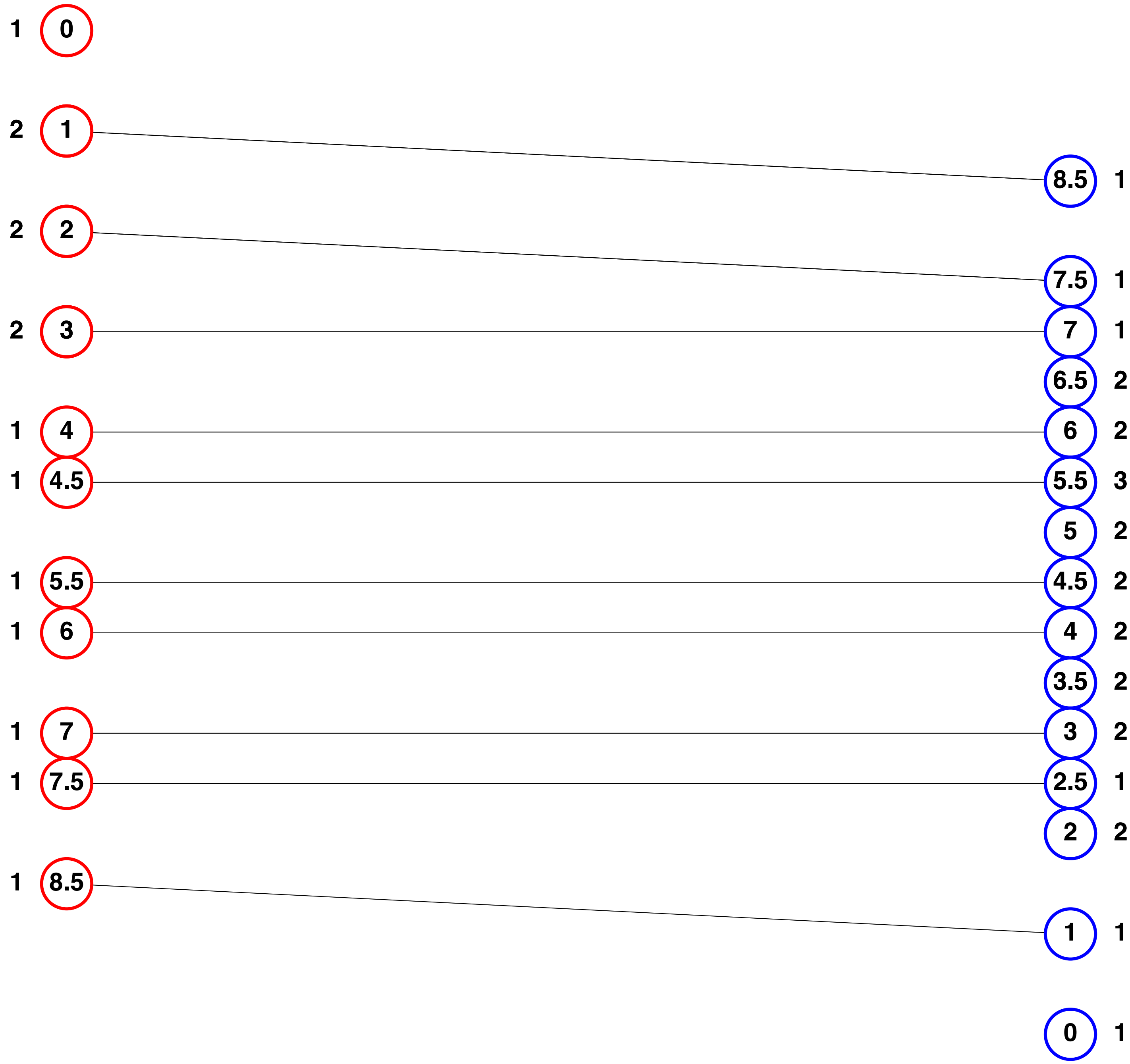
2 2
1 1
0 1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2 3
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6 2
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5 2
1 5.5
4.5 2
1 6
4 2
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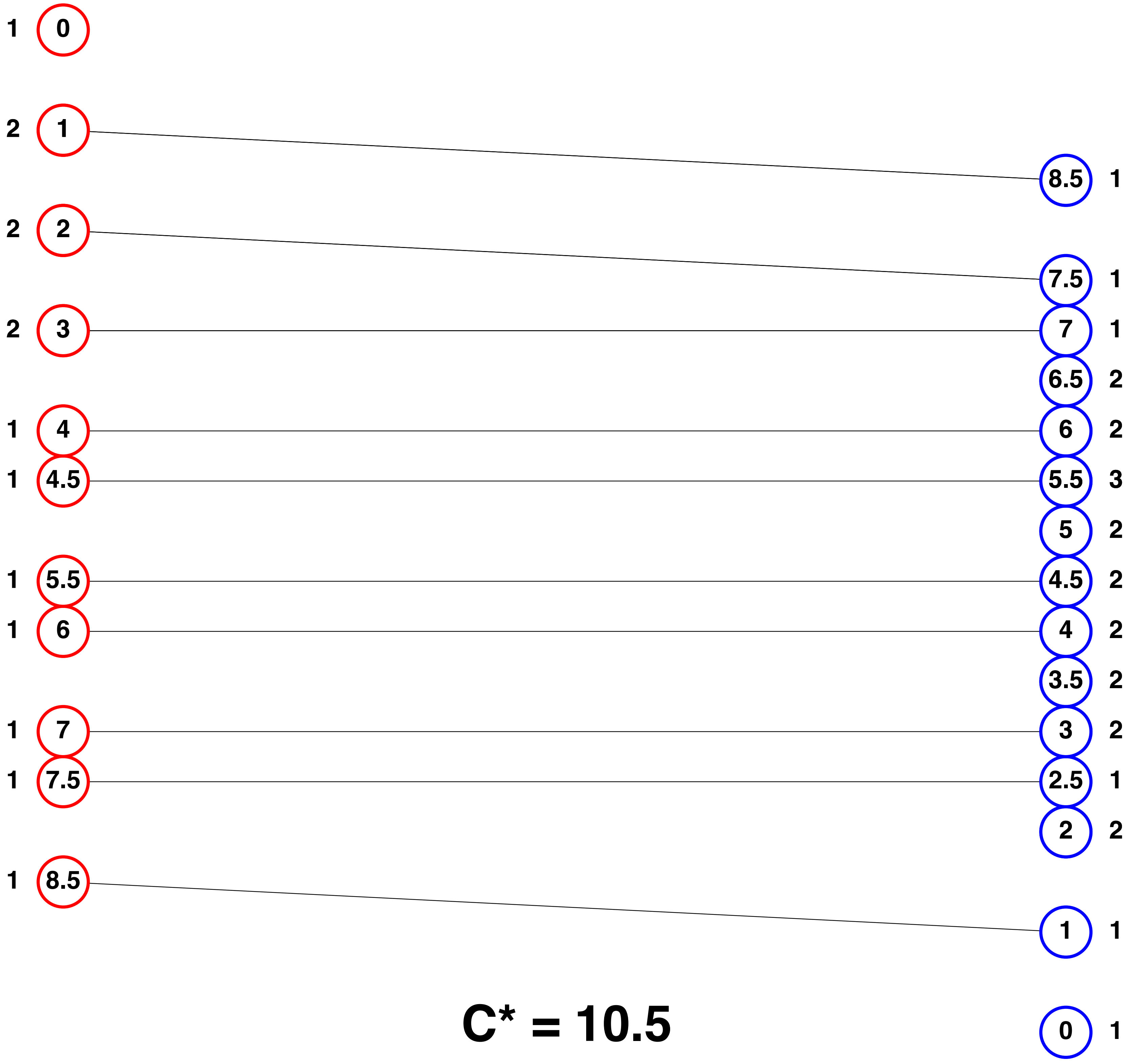
1 7
3 2
1 7.5
2.5 1
2 2
1 8.5
1 1
0 1

$$C^* = 10.5$$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4.5
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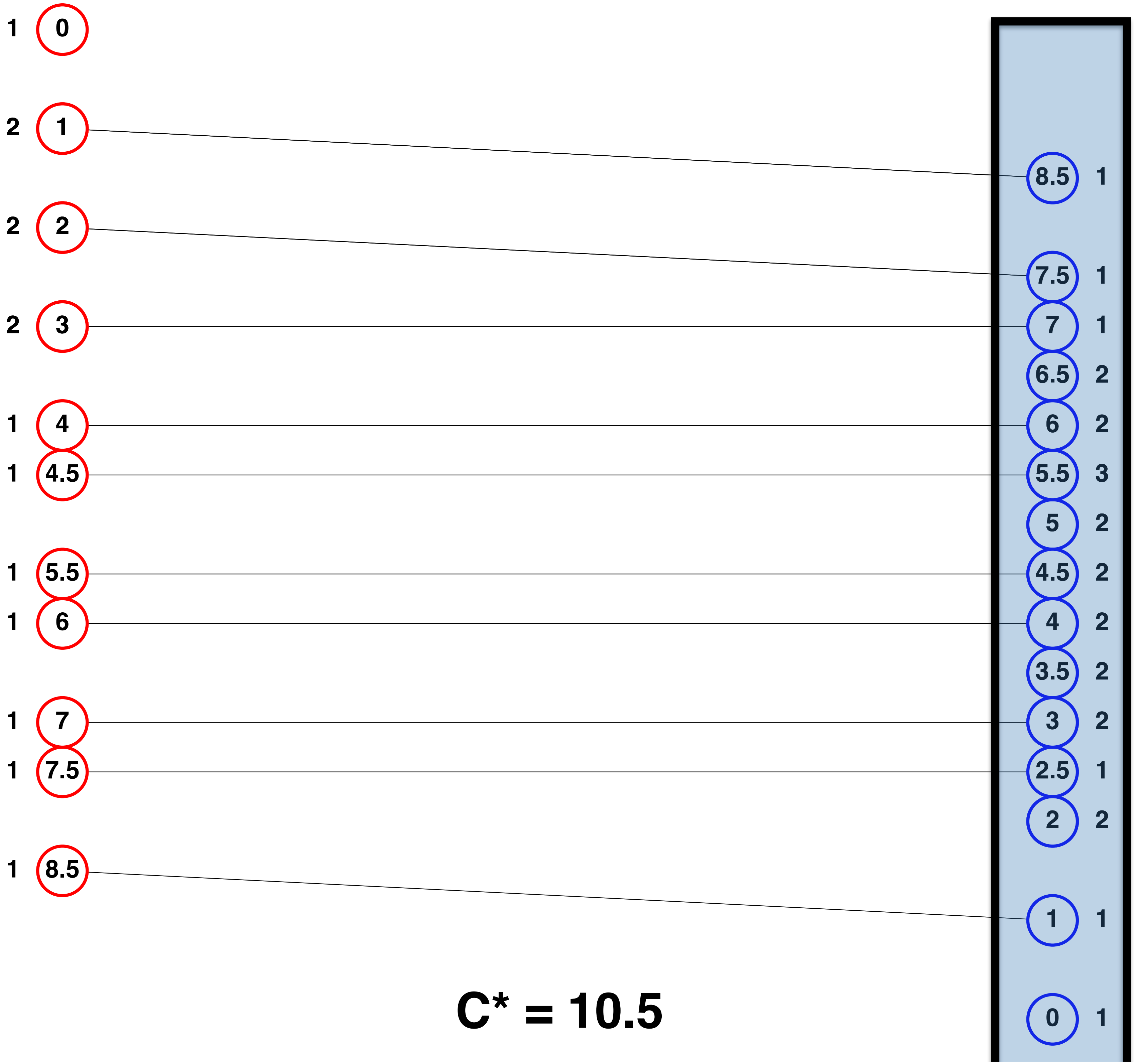
1

$C^* = 10.5$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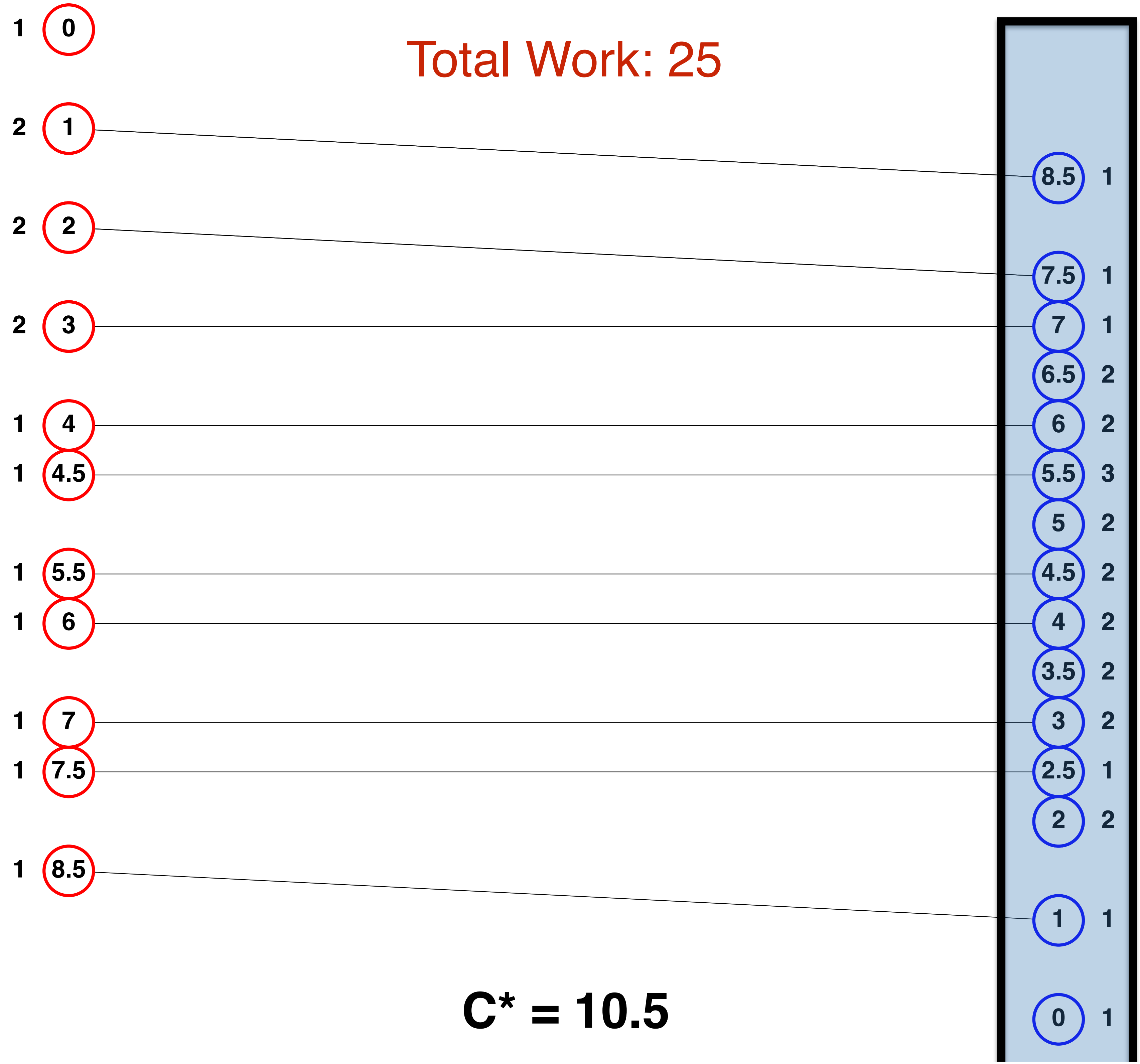
$C^* = 10.5$



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7 1
6.5 2
6 2
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4 2
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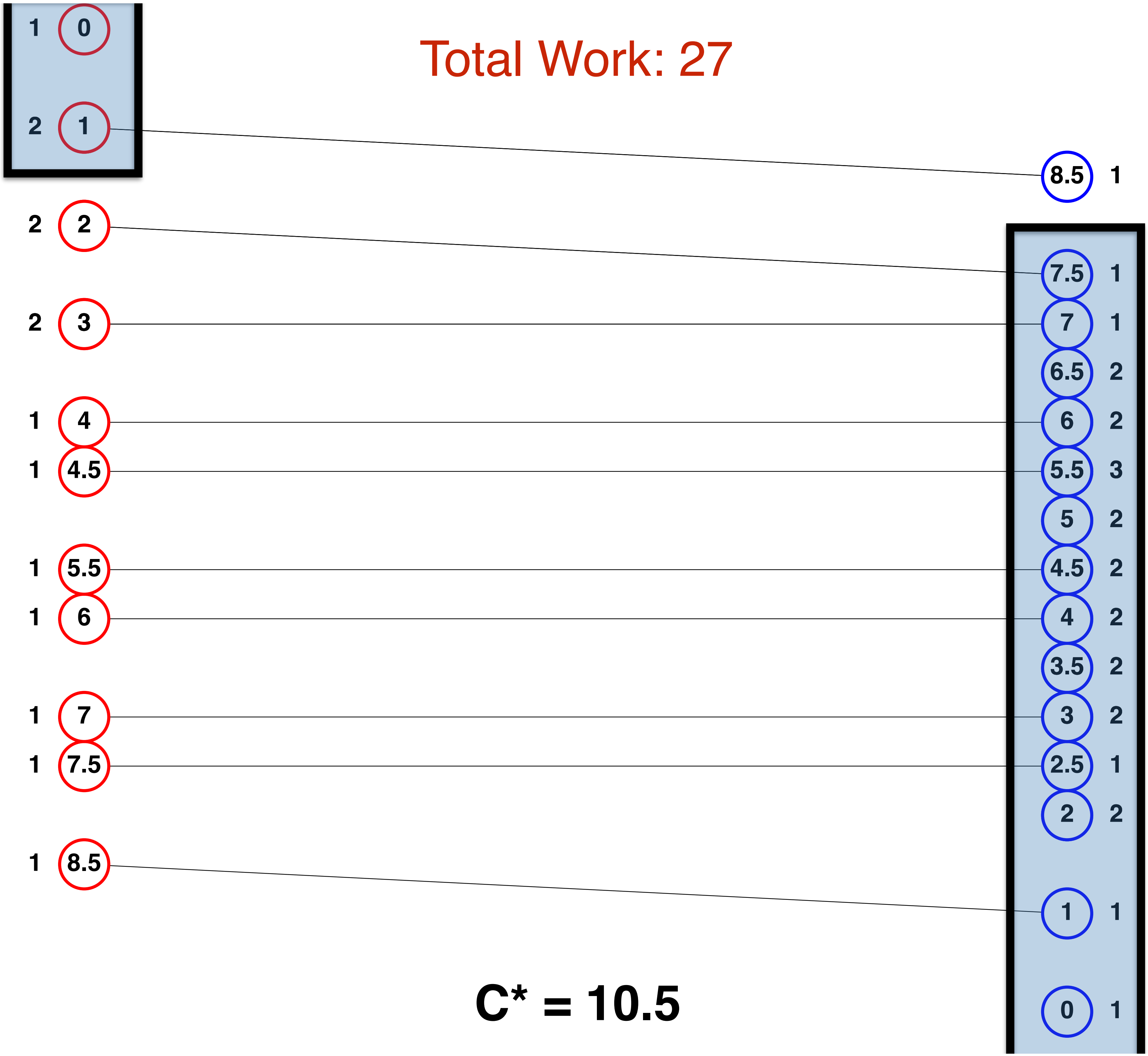
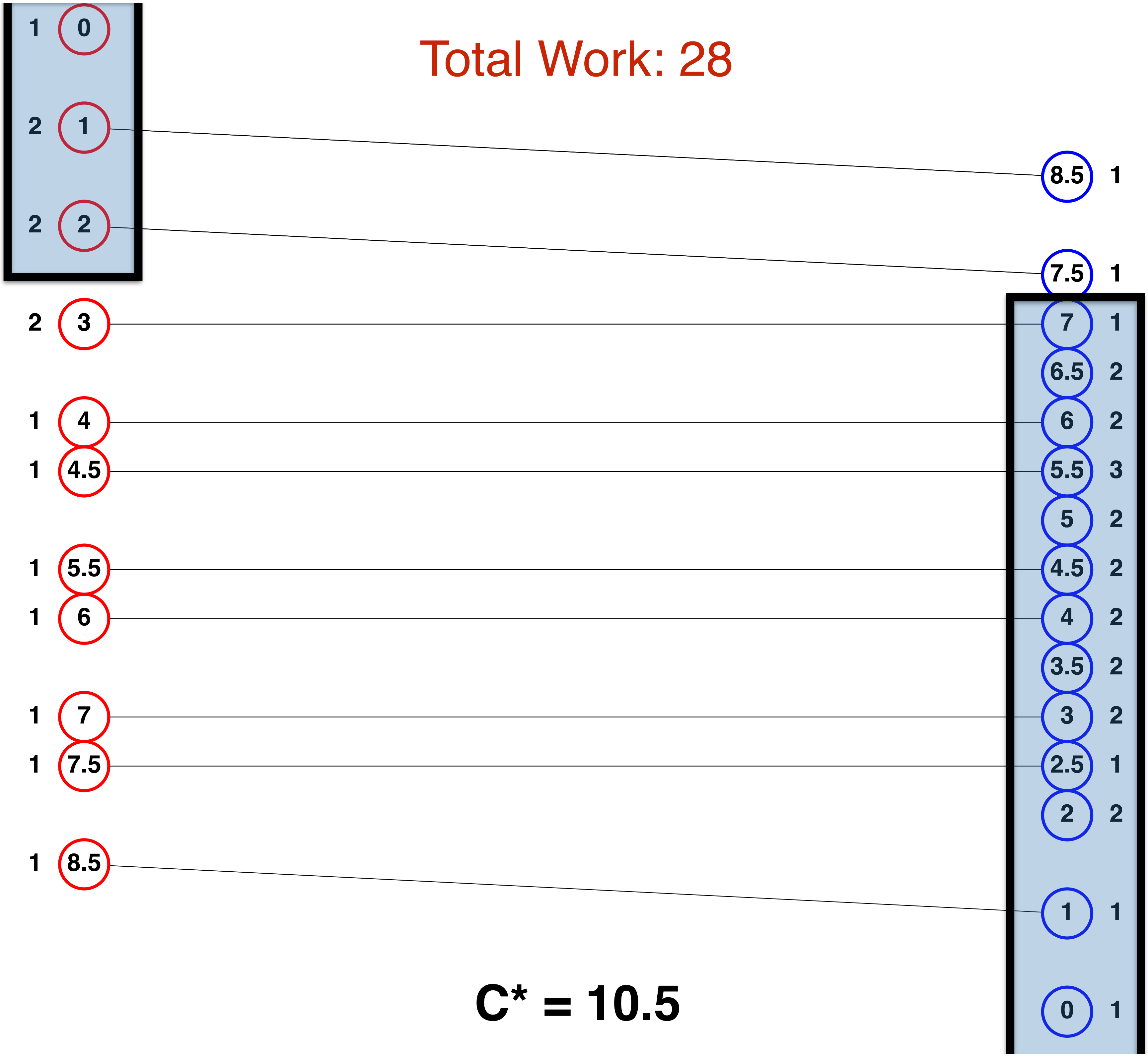
3 2
2.5 1
2 2
1 1
0 1
C* = 10.5

Total Work: 27



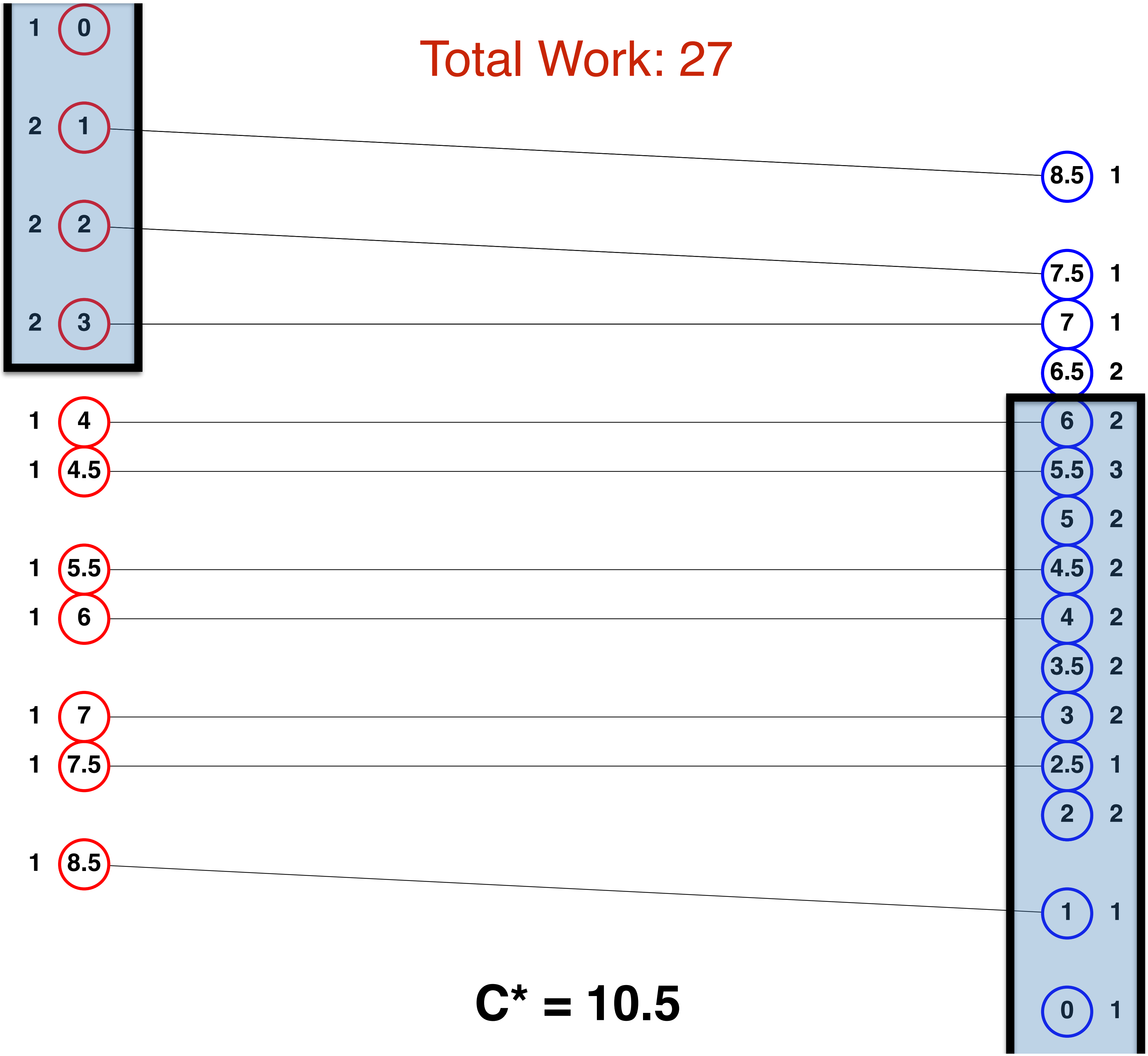
$C^* = 10.5$

Total Work: 28



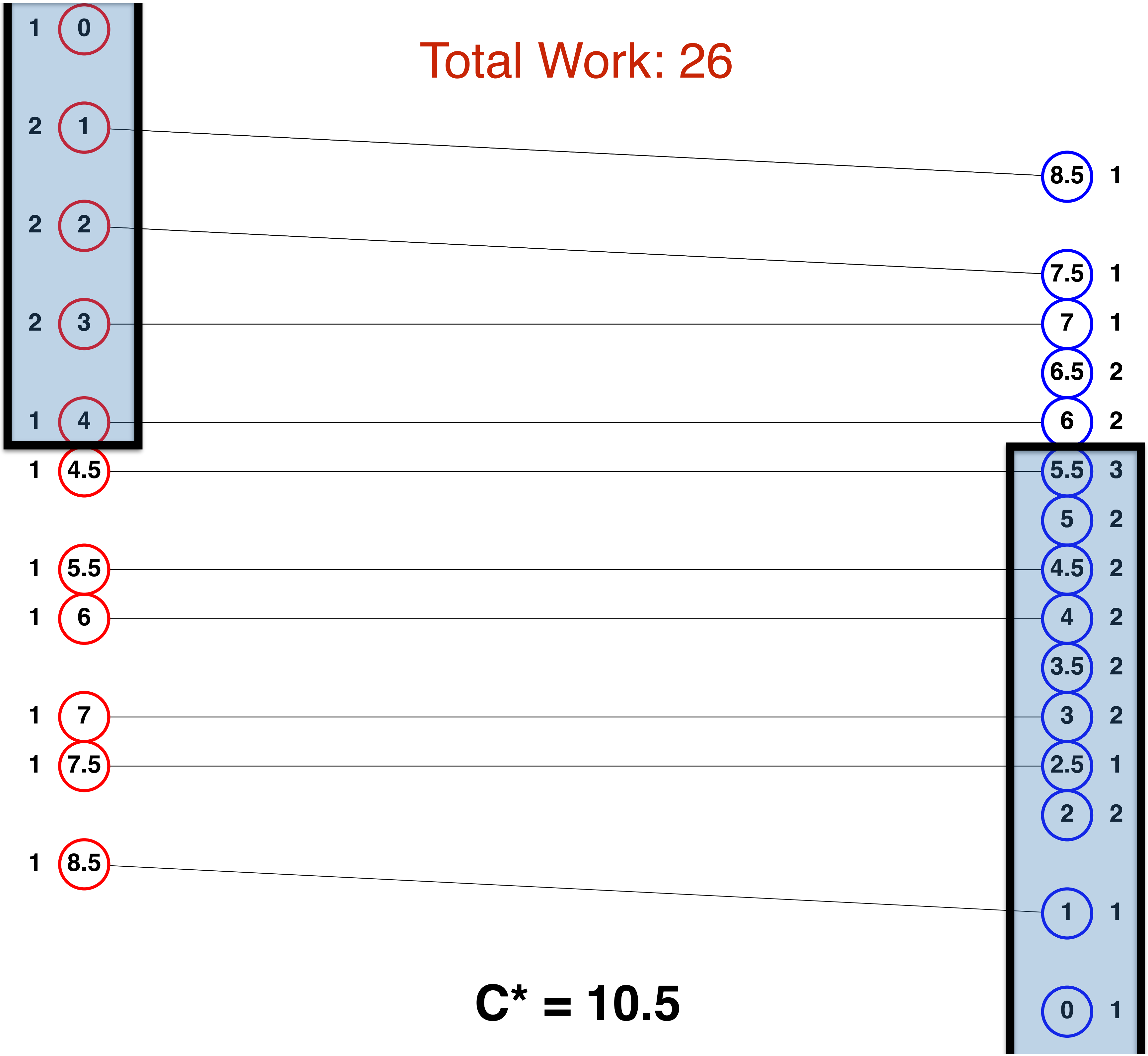
$C^* = 10.5$

Total Work: 27



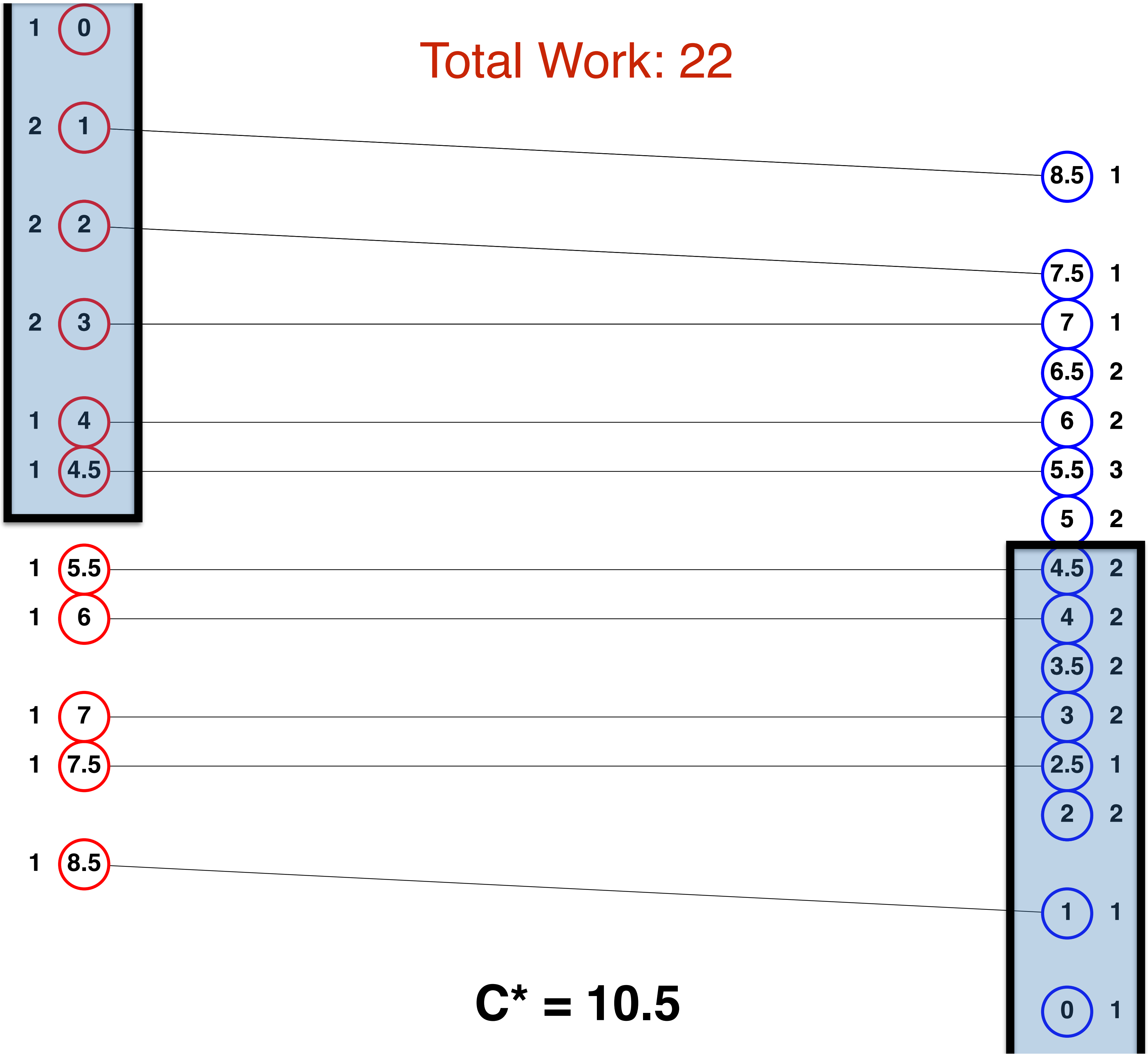
$C^* = 10.5$

Total Work: 26



$C^* = 10.5$

Total Work: 22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1 7
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1 8.5
8.5 1
7.5 1
7 1
6.5 2
6 2
5.5 3
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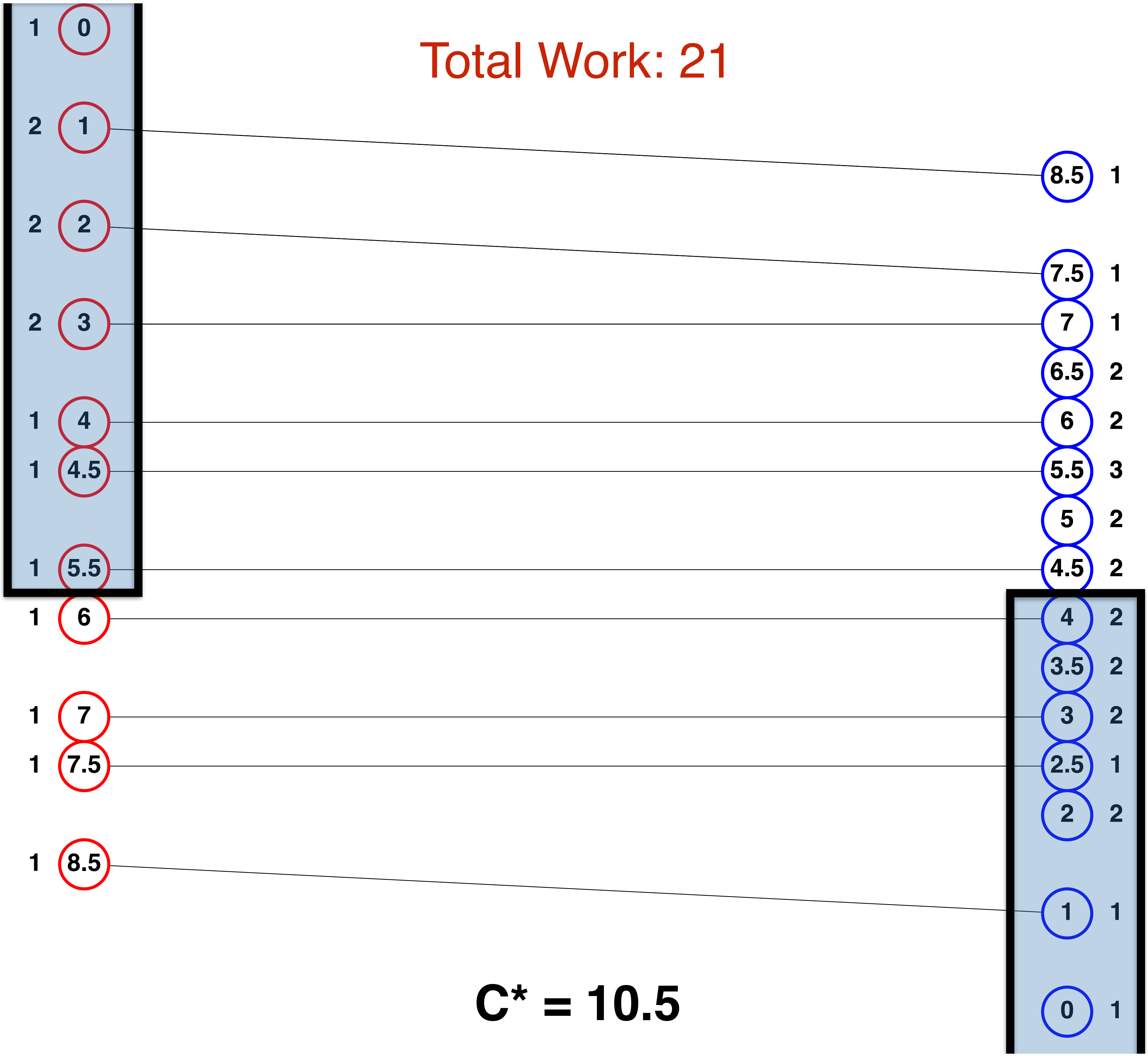
1 1
0 1

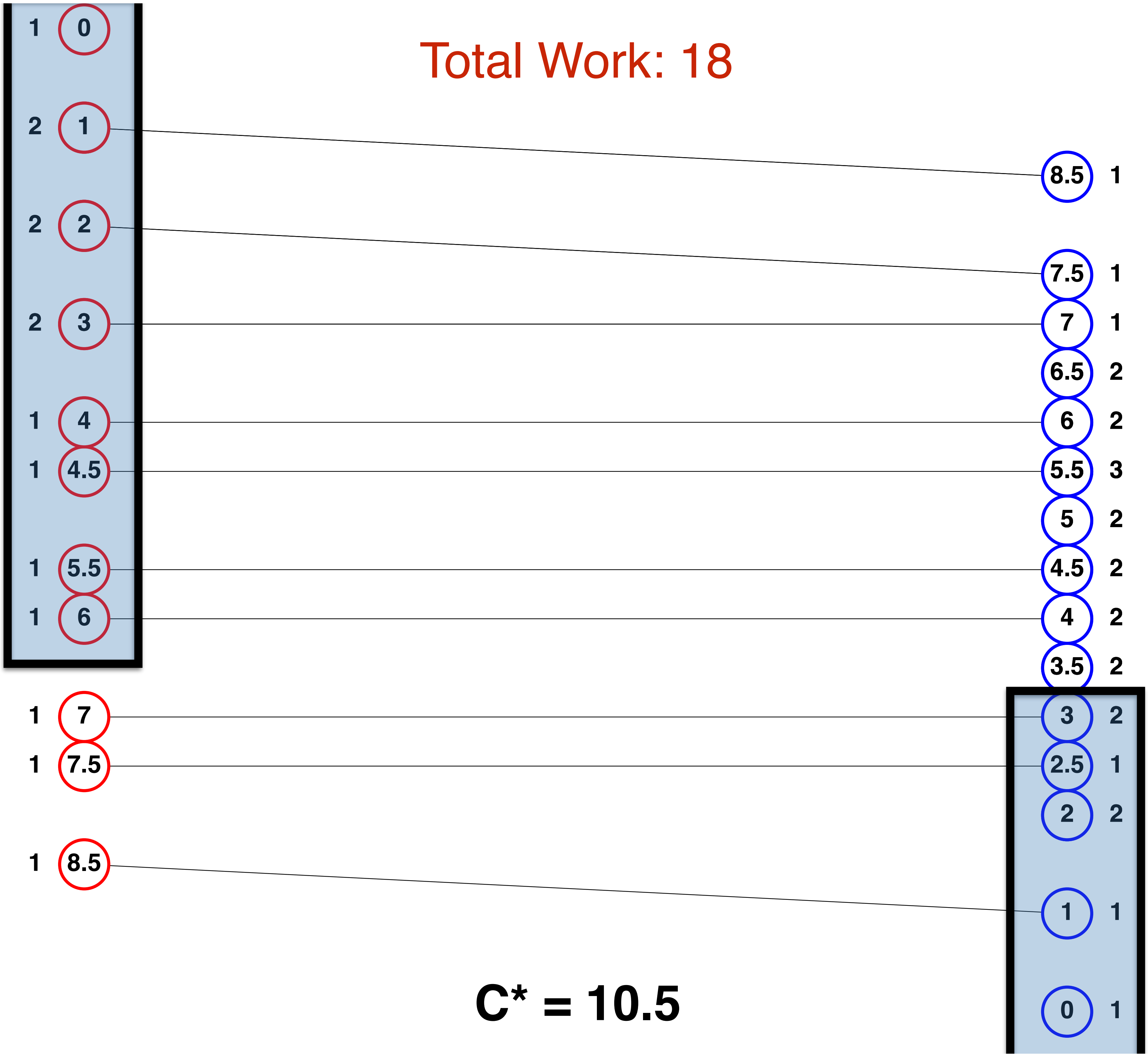
$C^* = 10.5$

Total Work: 21

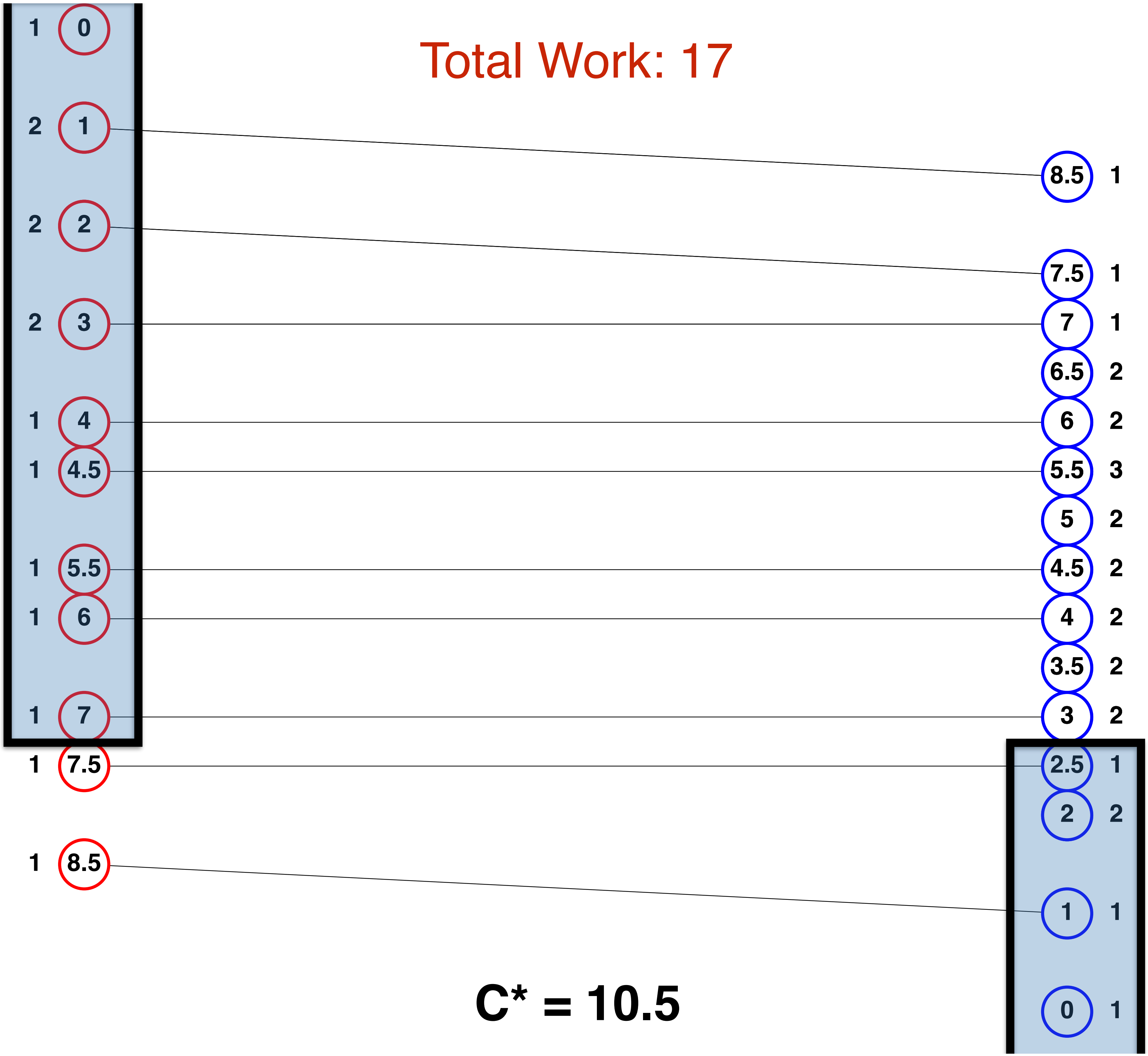


$$C^* = 10.5$$

Total Work: 18



$C^* = 10.5$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7 1
6.5 2
6 2
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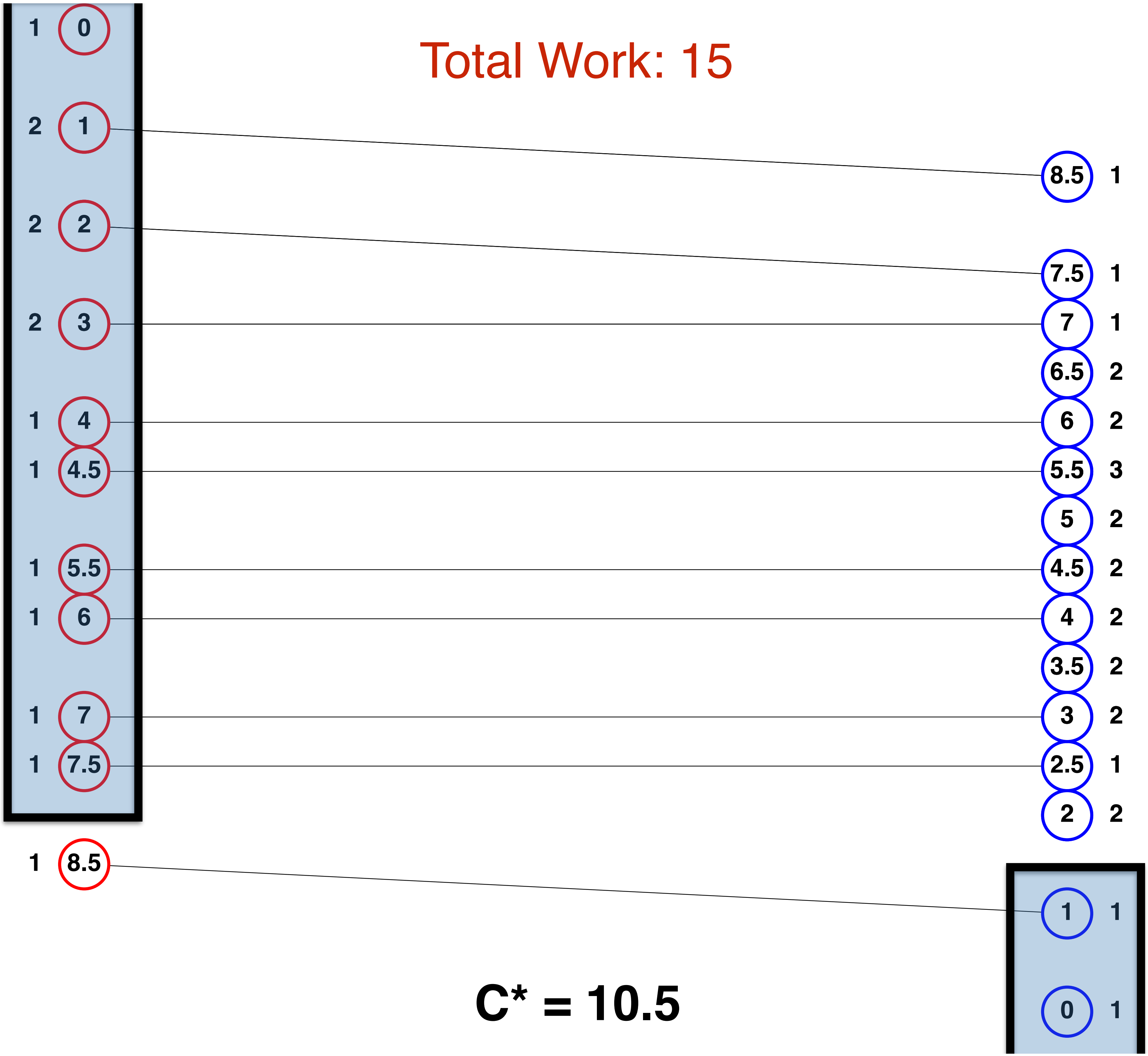
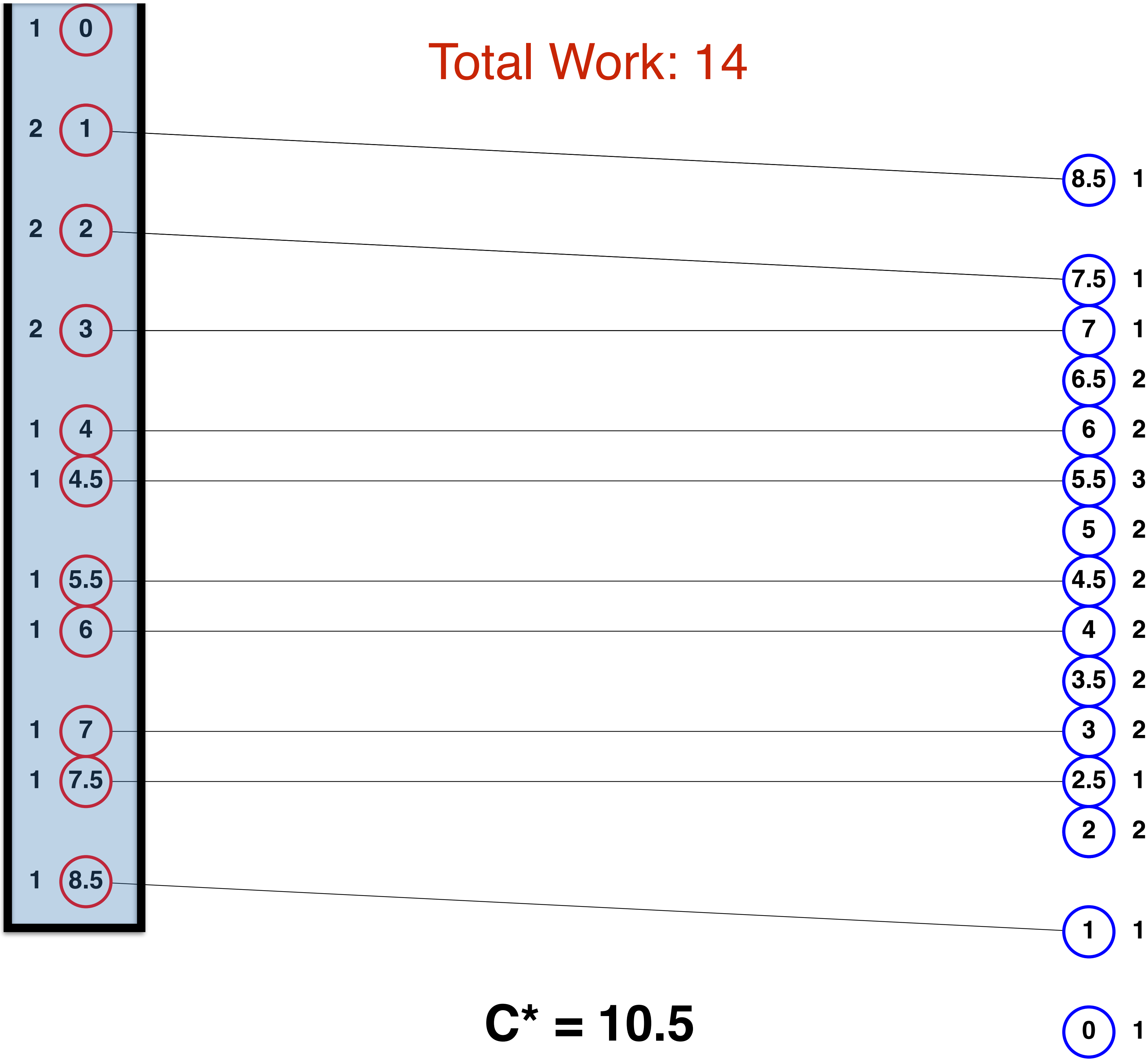
C* = 10.5

Total Work: 15

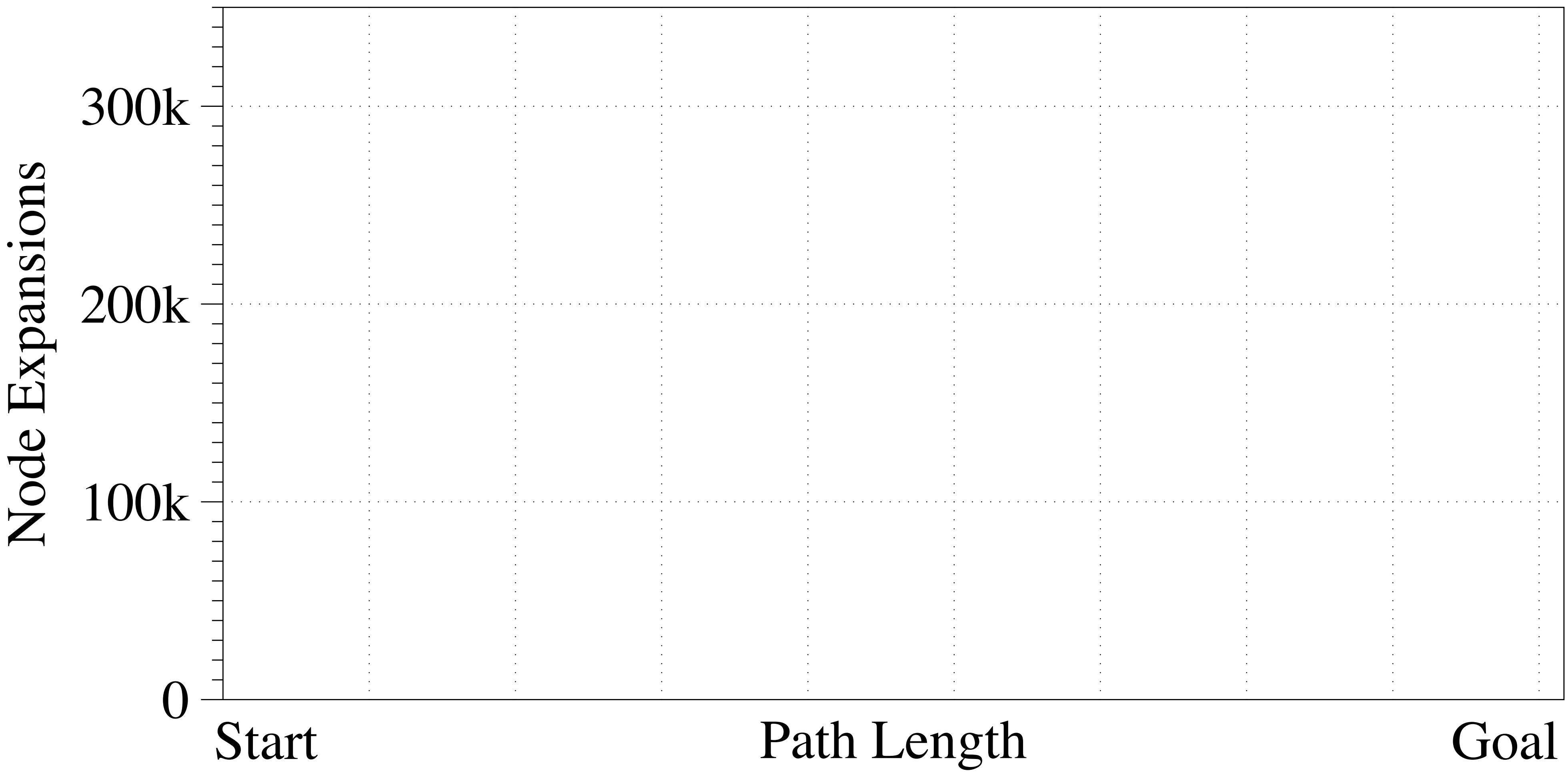


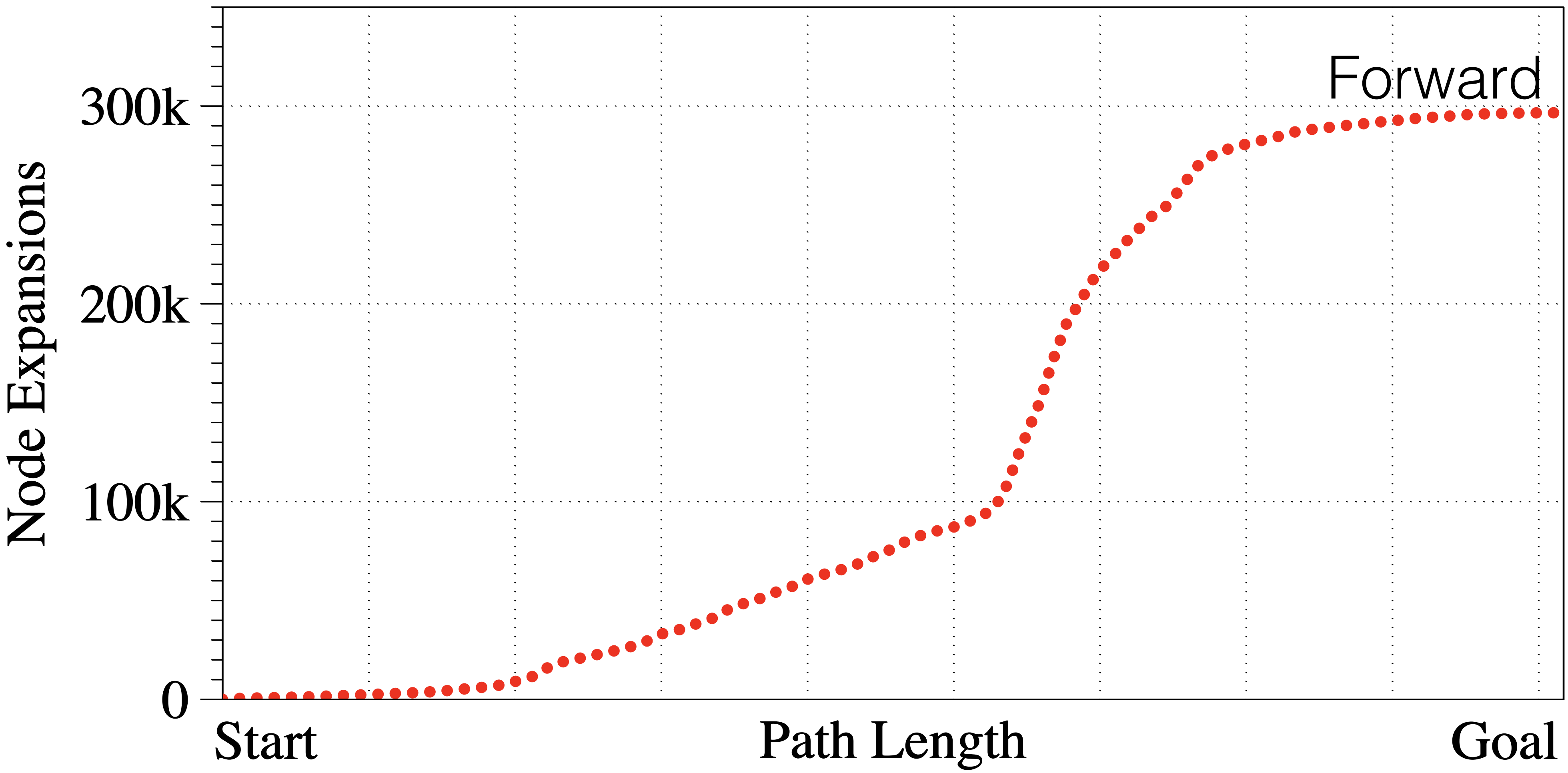
C* = 10.5

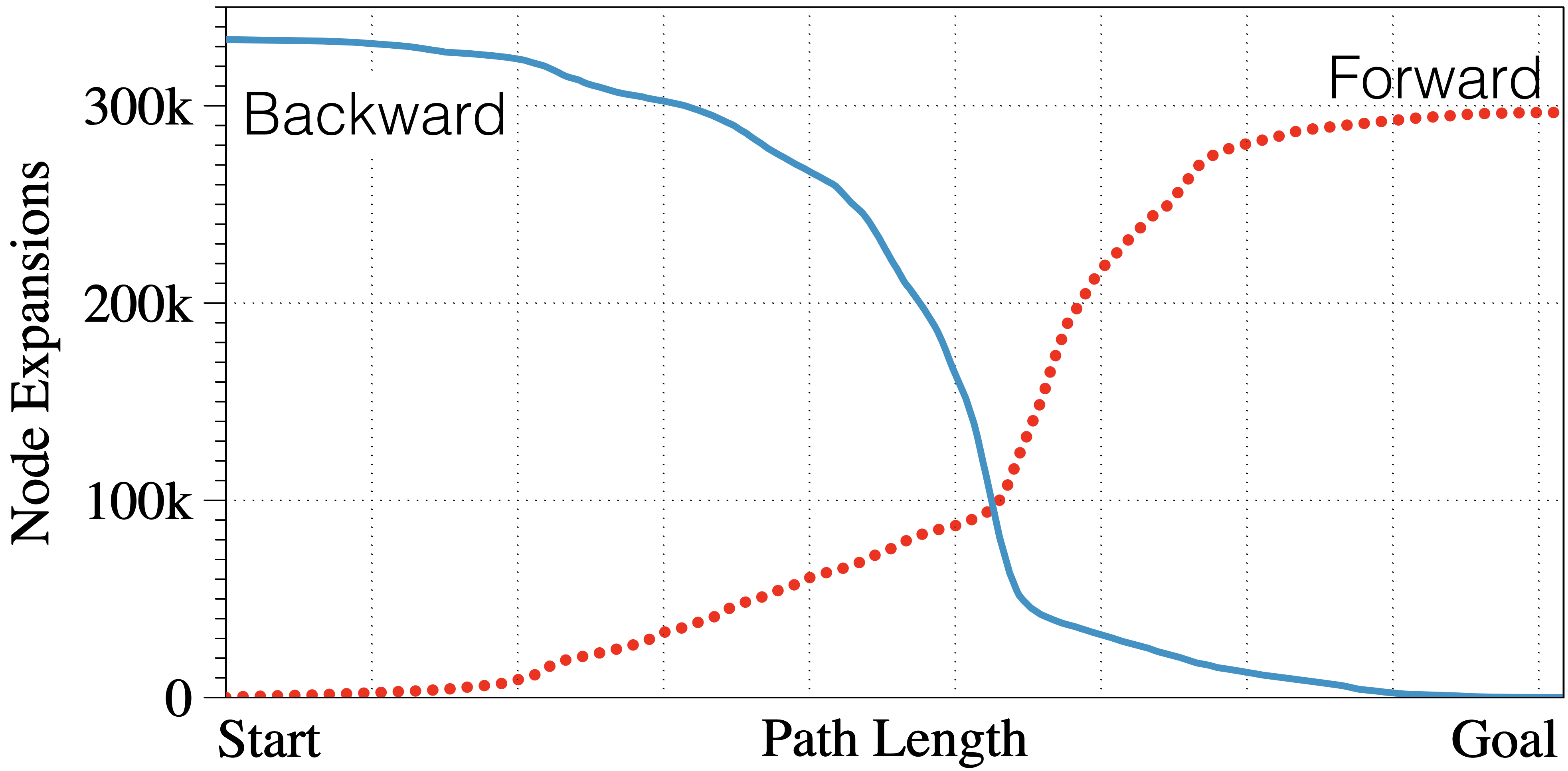
Total Work: 14



$$C^* = 10.5$$


Node Expansions
300k
200k
100k
0
Start
Path Length
Goal


Forward
Node Expansions
300k
200k
100k
0
Start
Path Length
Goal



Node Expansions
300k
200k
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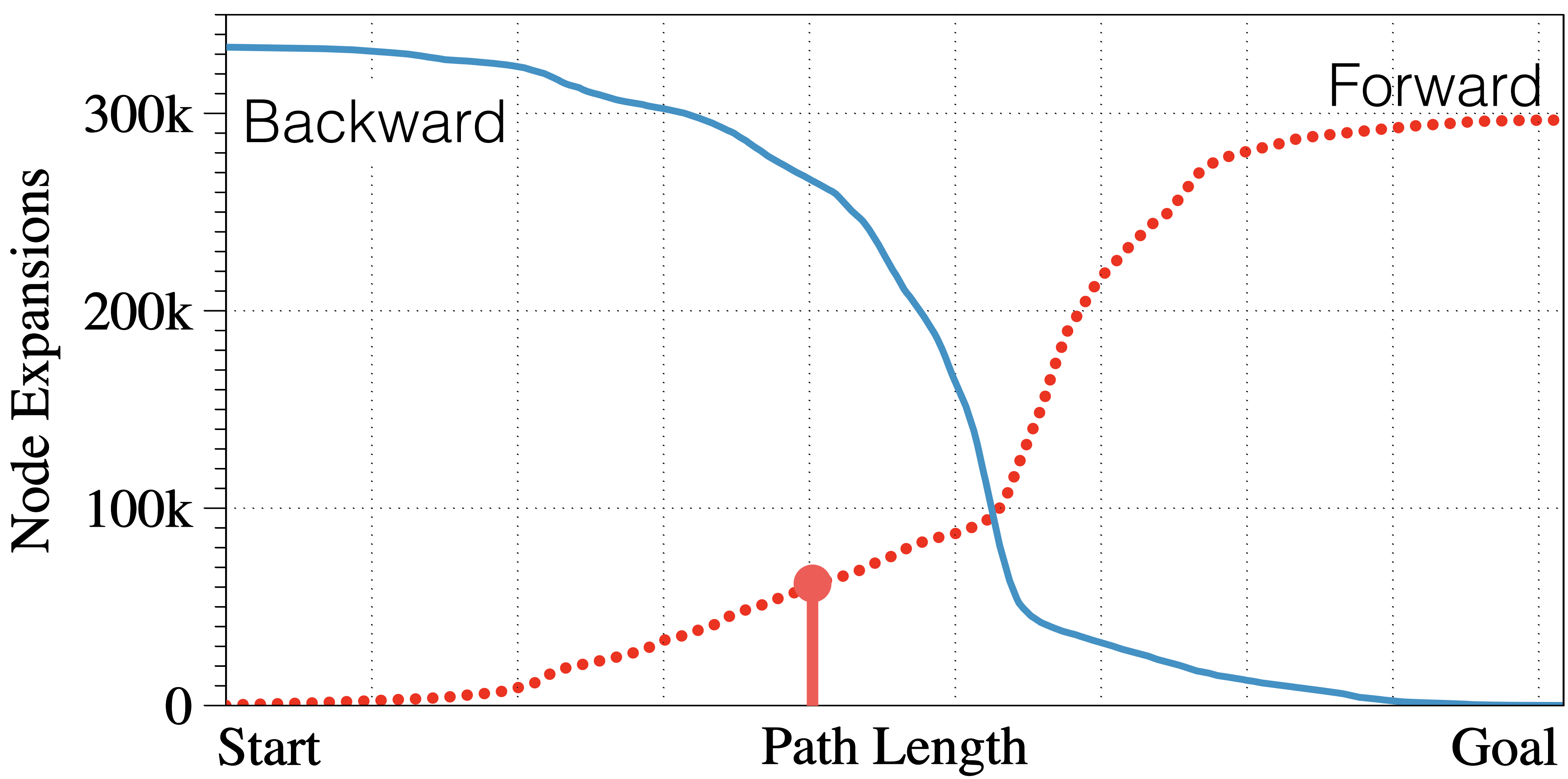
0
Backward
Forward
Start
Path Length
Goal



Node Expansions
300k
200k
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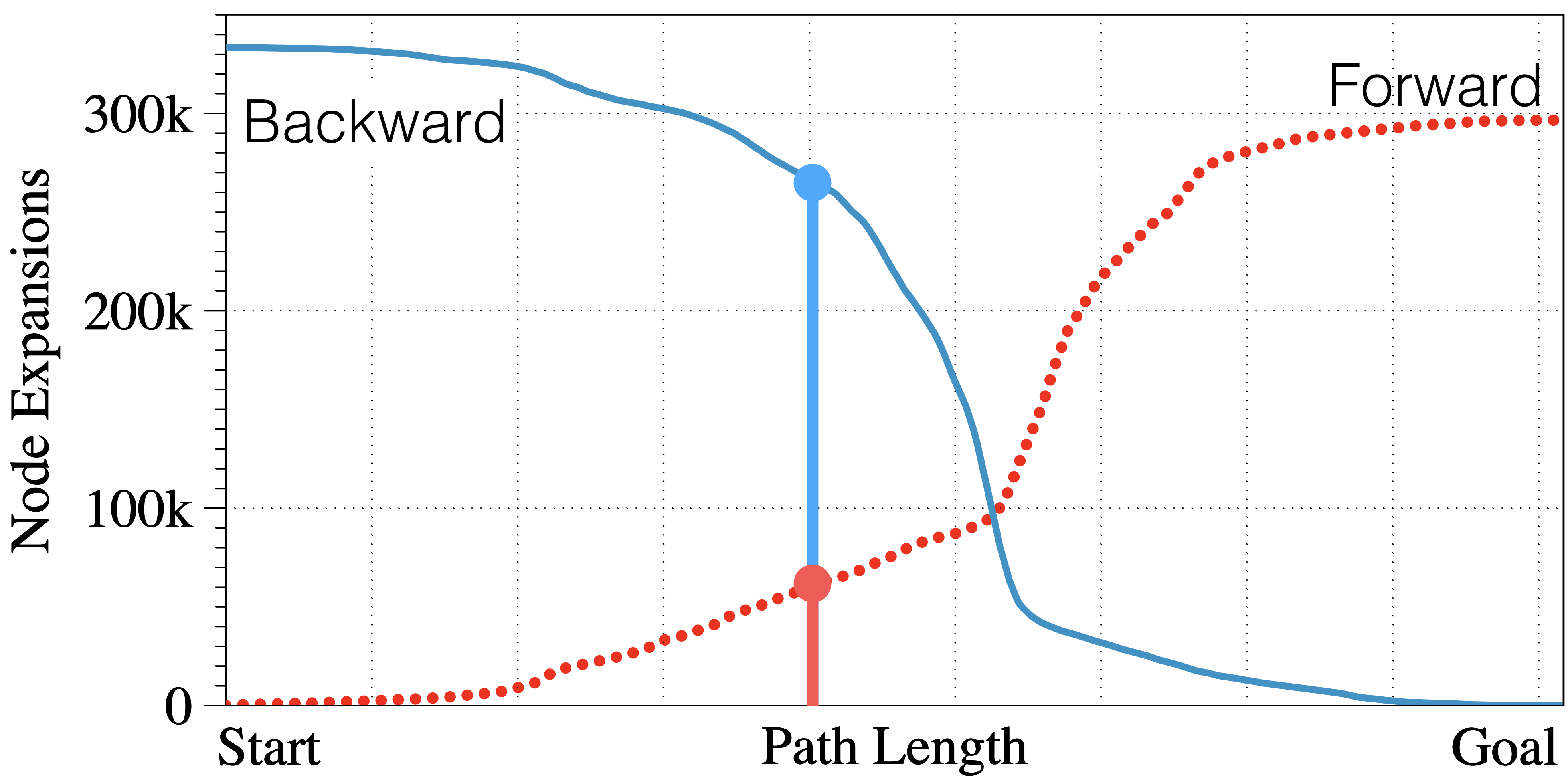
Backward
Forward
Start
Path Length
Goal



Node Expansions
300k
200k
100k
0
Backward
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Start
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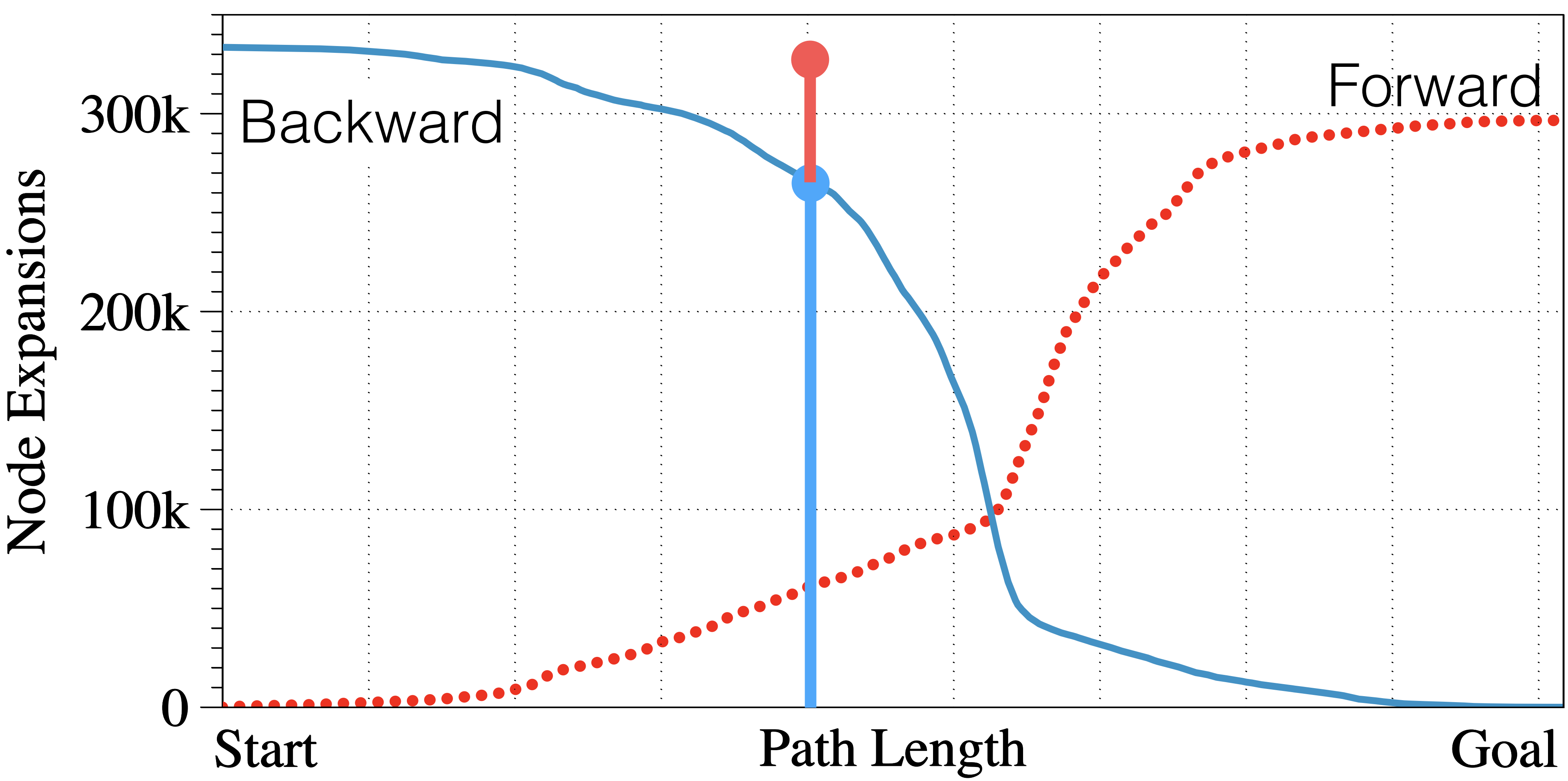

Node Expansions
300k
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100k
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Backward
Forward
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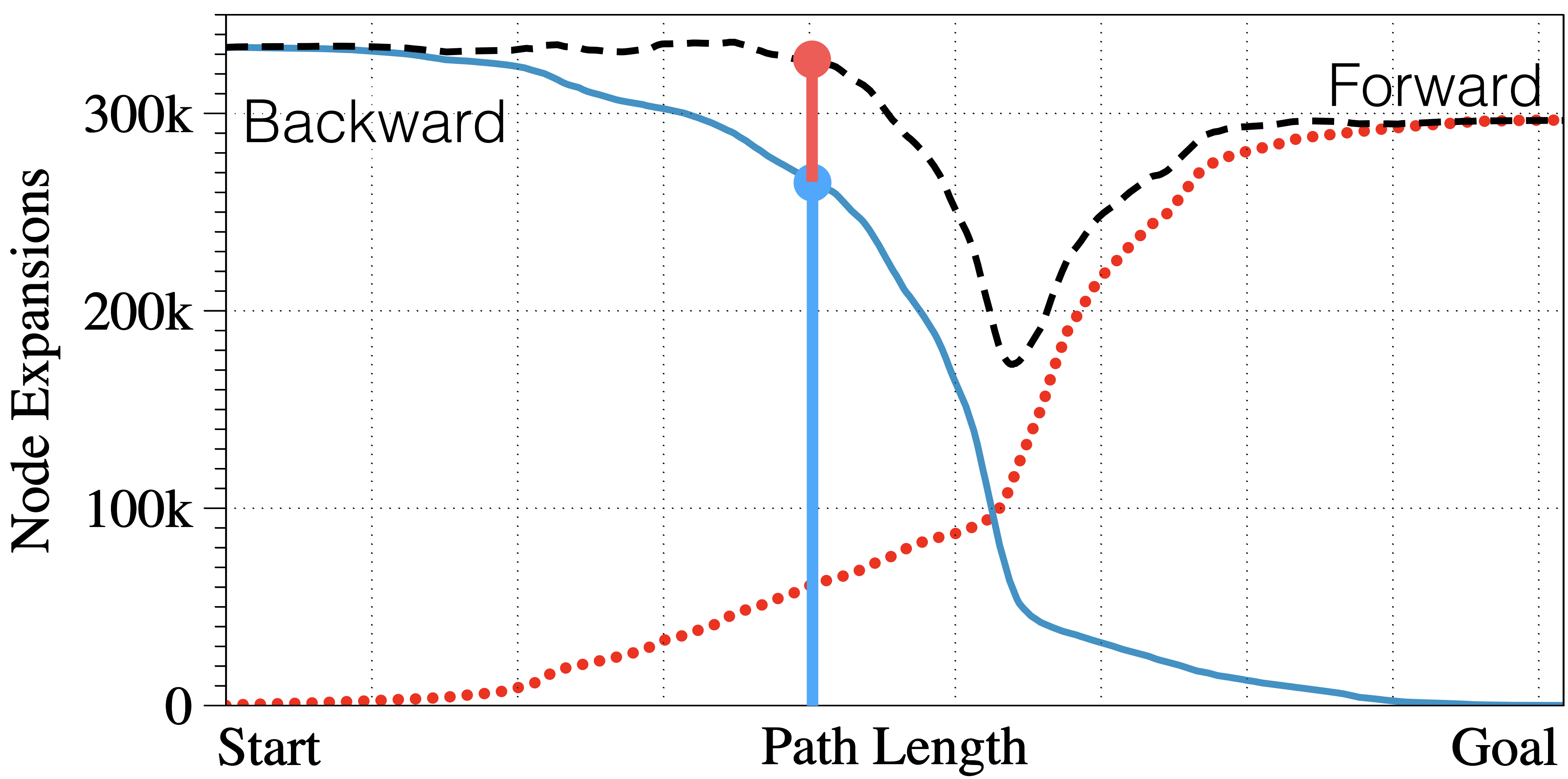
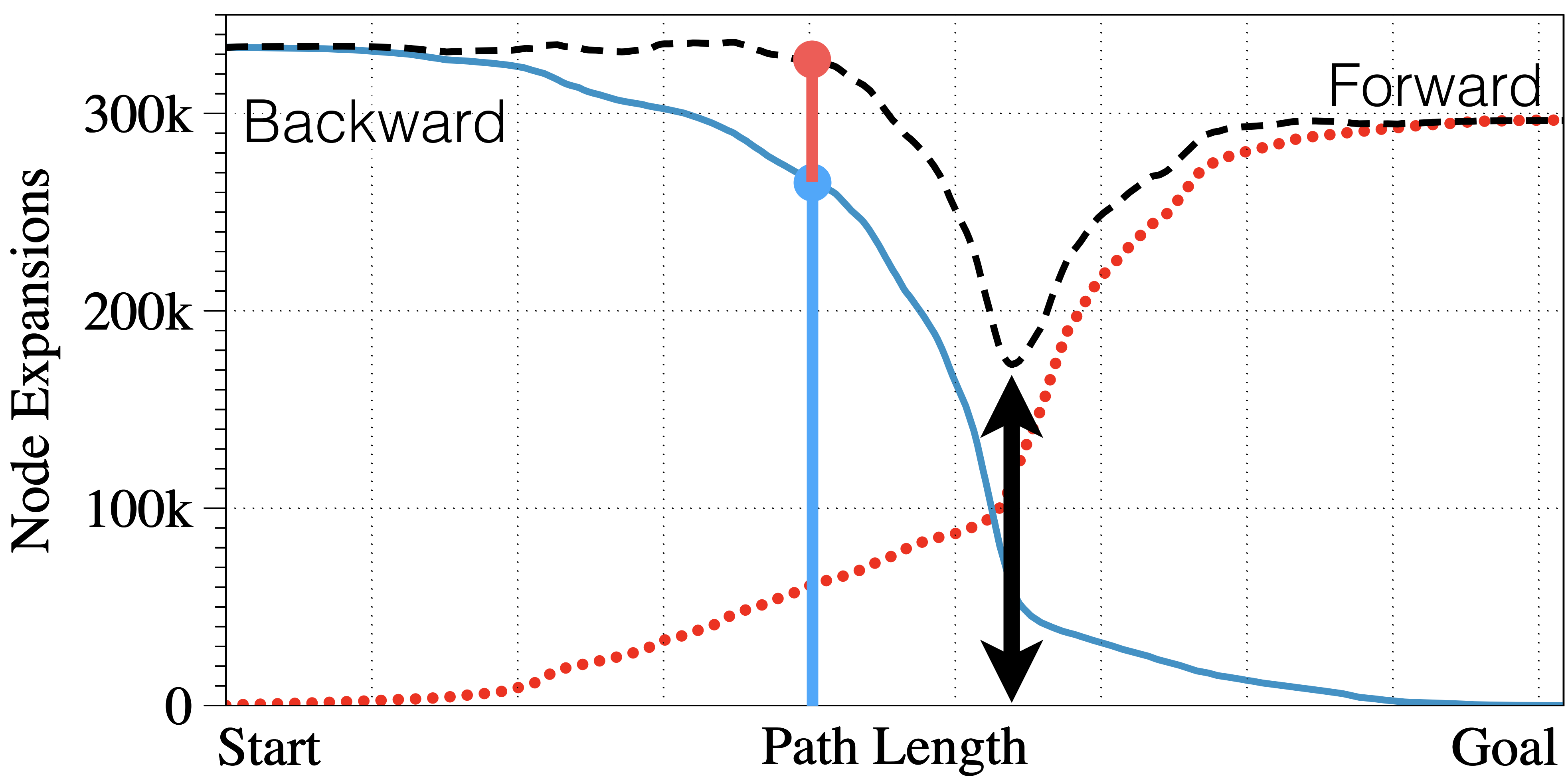
Path Length
Goal


Node Expansions
300k
200k
100k
0
Backward
Forward
Start
Path Length
Goal



Node Expansions
300k
200k
100k
0
Backward
Forward
Start
Path Length
Goal

Fractional MM

- Takes a parameter f
  - Cost of the state space to explore in each direction
  - Costs correspond to different vertex covers

- *We can (offline) compute the best algorithm for a given search problem*

The Minimal Set of States that Must be Expanded in a Front-to-end Bidirectional Search,
Eshed Shaham, Ariel Felner, Jingwei Chen and Nathan R. Sturtevant,
Symposium on Combinatorial Search (SoCS), **2017**

Vertex Cover on a Bipartite Graph

- Approximation algorithm:
  - Repeat until all vertices covered
    - Choose any edge/line with uncovered vertex
    - Place both states into vertex cover

- Gives 2x approximation to optimal vertex cover
  - (Papadimitriou & Steiglitz, 1982)

Using this algorithm

- We don’t know the full graph ahead of time

Using this algorithm

- We don't know the full graph ahead of time
  - Build the graph as we go

Using this algorithm

- We don’t know the full graph ahead of time
  - Build the graph as we go
- We don’t know the optimal solution cost

Using this algorithm

- We don't know the full graph ahead of time
  - Build the graph as we go
- We don't know the optimal solution cost
  - Must estimate C*

Using this algorithm

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Using this algorithm

- We don’t know the full graph ahead of time
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  - Must estimate C*
- We must avoid re-expanding states
  - Carefully order state expansions

Using this algorithm

- We don’t know the full graph ahead of time
  - Build the graph as we go
- We don’t know the optimal solution cost
  - Must estimate C*
- We must avoid re-expanding states
  - Carefully order state expansions
- Computing lb(*u*, *v*) could be expensive

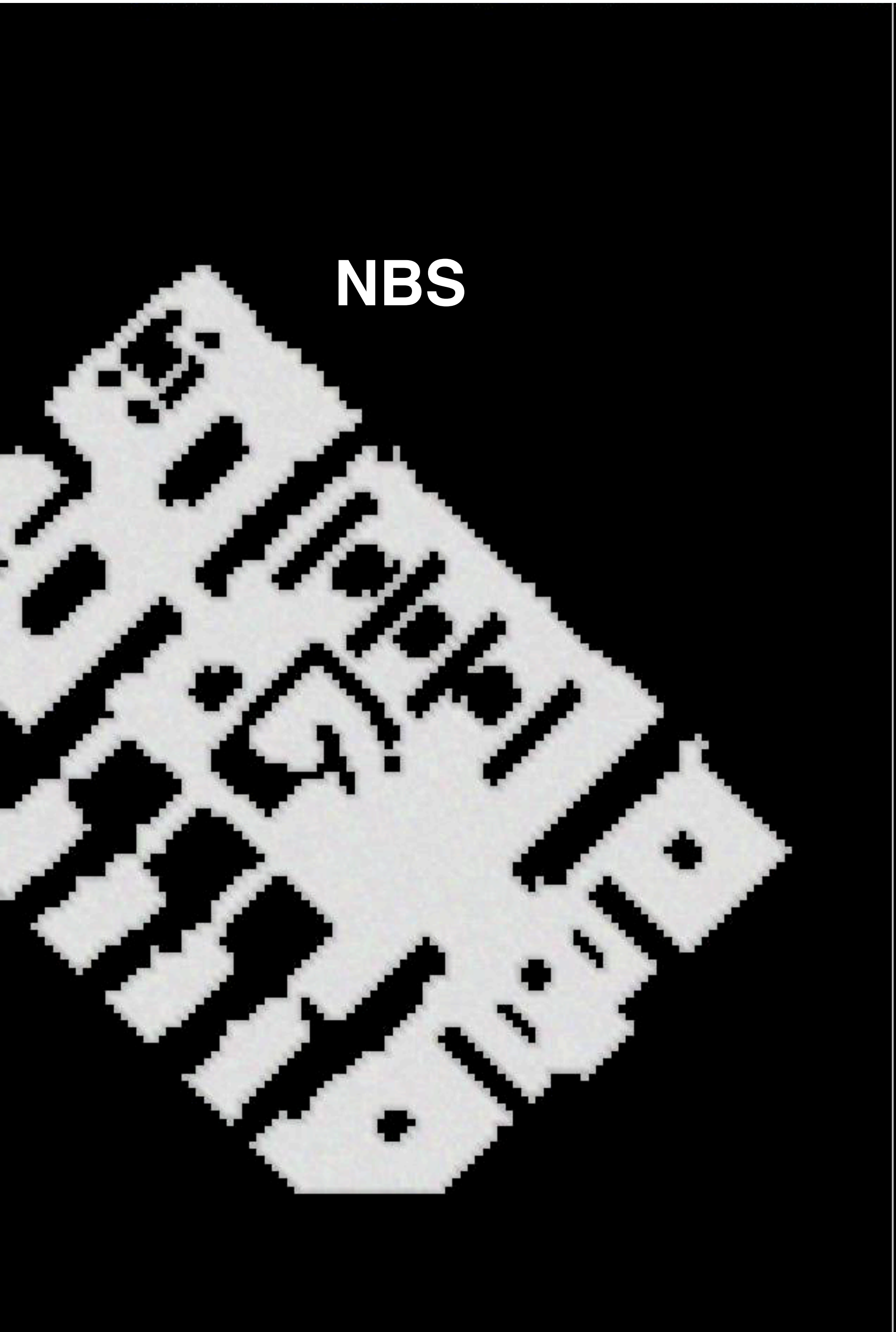
Using this algorithm

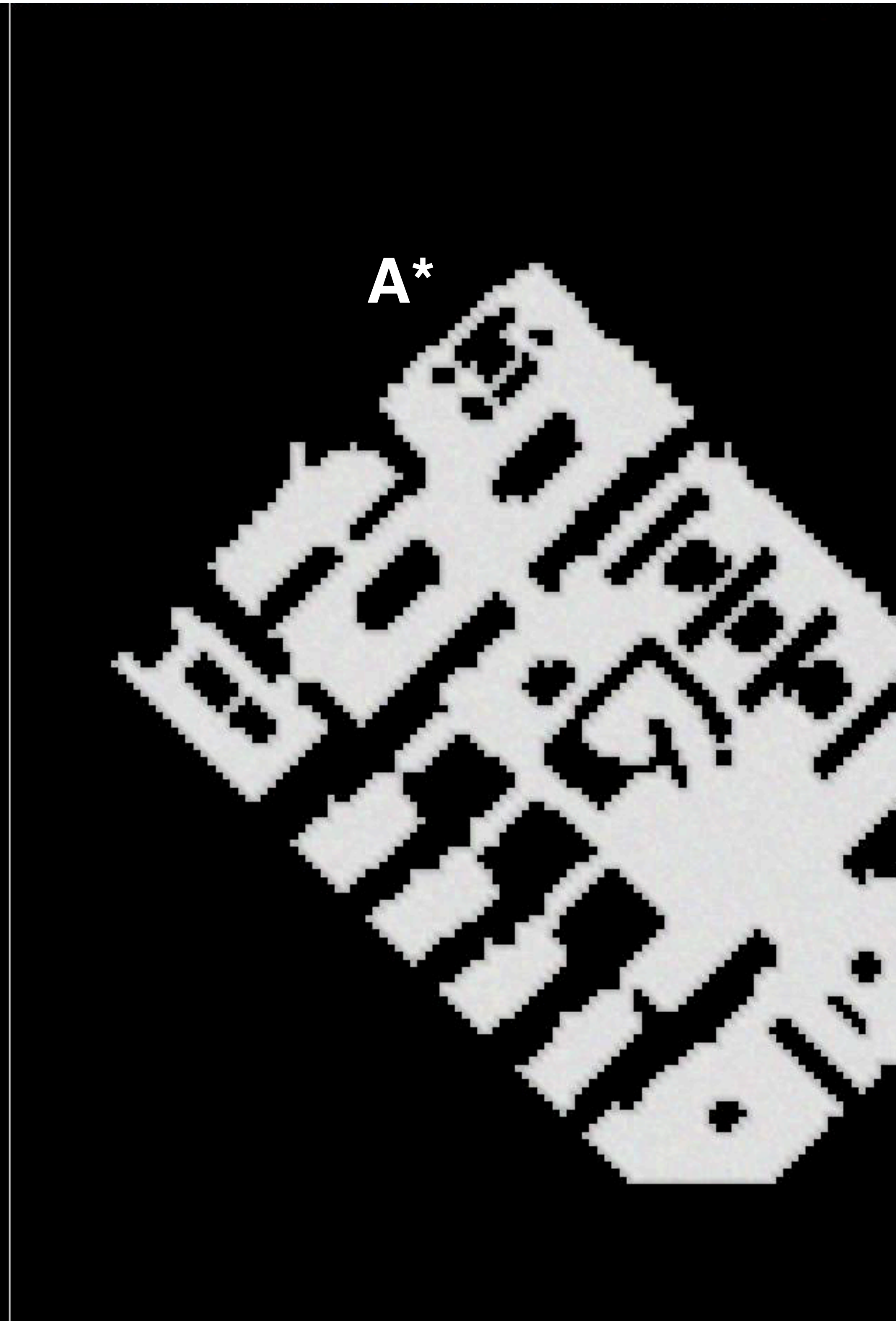
- We don’t know the full graph ahead of time
  - Build the graph as we go
- We don’t know the optimal solution cost
  - Must estimate C*
- We must avoid re-expanding states
  - Carefully order state expansions
- Computing lb(*u*, *v*) could be expensive
  - Efficient data structures

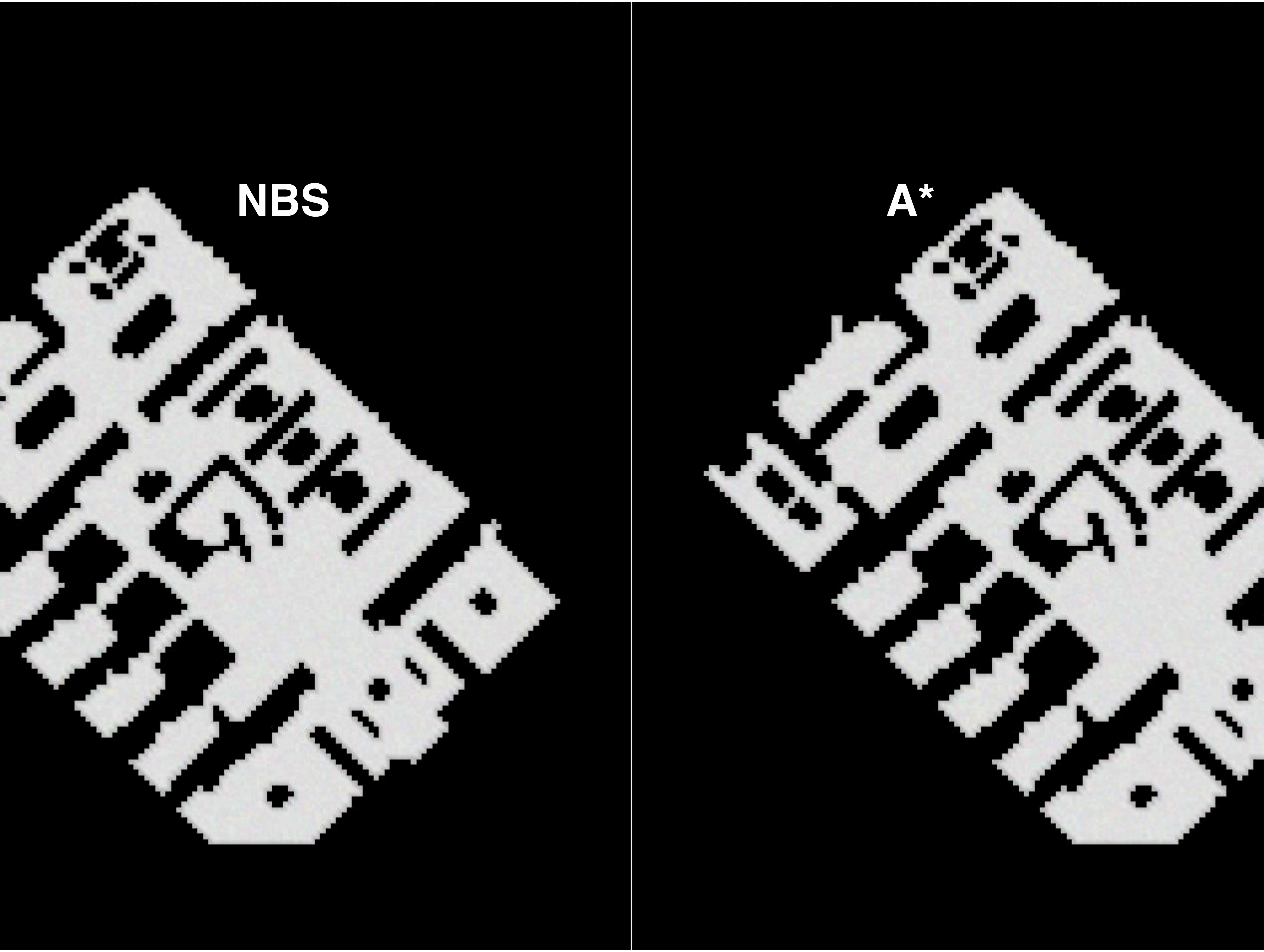
NBS

- Put start/goal onto forward/backward priority queues
- While forward/backward not empty
  - Among all state on queues:
    - Select the pair with lowest *lb*
    - Expand **both** of them
  - Terminate when *lb* ≥ best path
- Gives 2x bound on optimal number of expansions
  - Bound is tight

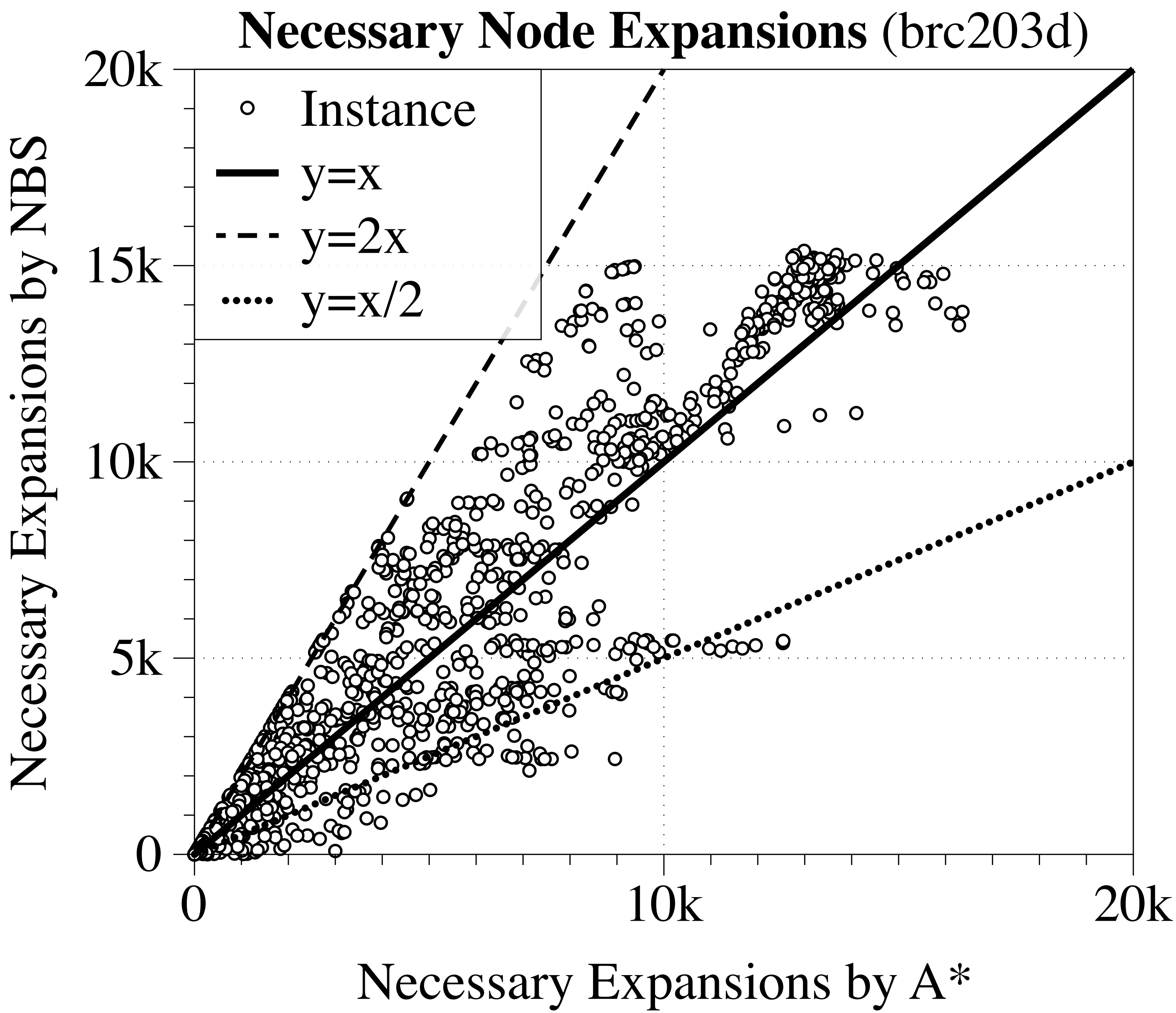
Front-to-End Bidirectional Heuristic Search with Near-Optimal Node Expansions, Jingwei Chen, Robert C. Holte, Sandra Zilles and Nathan R. Sturtevant, International Joint Conference on Artificial Intelligence (IJCAI), **2017**


NBS

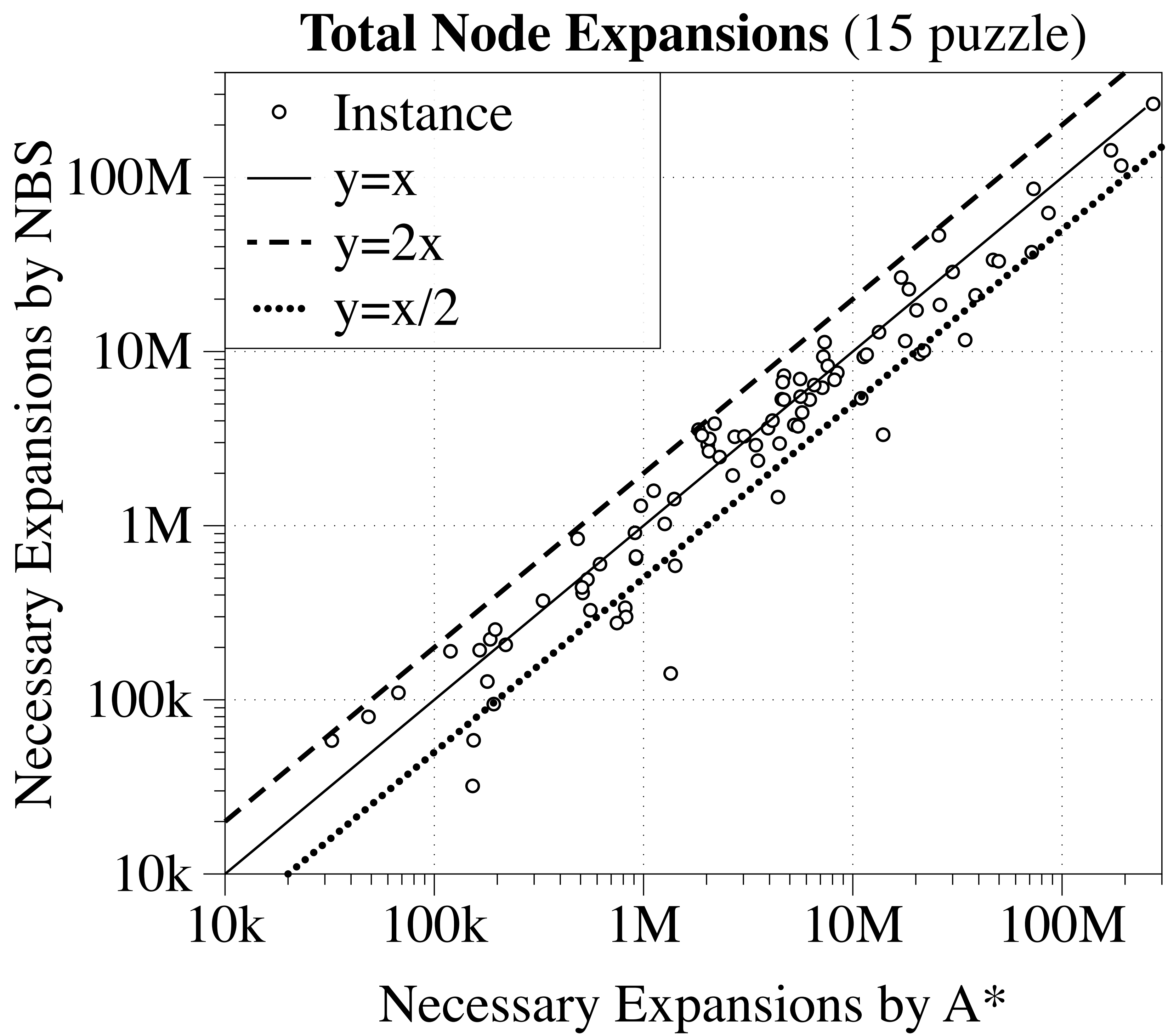

A*



NBS
A*



Necessary Node Expansions (brc203d)
Instance
y=x
y=2x
y=x/2
Necessary Expansions by NBS
Necessary Expansions by A*
0
5k
10k
15k
20k
0
10k
20k


Total Node Expansions (15 puzzle)
Instance
y=x
y=2x
y=x/2
Necessary Expansions by NBS
10k
100k
1M
10M
100M
Necessary Expansions by A*

Summary

- Theory
  - First definition of necessary node expansions
  - fMM - implements optimal bidirectional search
- Practice
  - Near-optimal approach (NBS)
  - Node expansions are bounded by 2x optimal
- Demos & videos will appear at:
  - https://www.movingai.com

Open Questions

- What can we learn about bidirectional search from the minimum vertex cover?
- Is there an algorithm with better average performance?
- Efficient front-to-front search?